Biseparable representations of the Certainty Equivalents

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Abstract

We study the following biseparable representation of the certainty equivalent:

$$F(x, y; p) = u^{-1}\big(w(p)u(x) + (1 - w(p))u(y)\big) \quad \text{for } x \geq y,\ p \in [0, 1],$$

where $(x, y; p)$ denotes a binary monetary prospect, $u : \mathbb{R} \to \mathbb{R}$ is a utility function, and $w : [0, 1] \to [0, 1]$ is a probability weighting function. We provide axiomatic characterizations of this representation over the full domain of binary prospects, as well as over a restricted domain of simple prospects, in which one of the payoffs is fixed. Our analysis covers three key cases: the general rank-dependent model, where w is arbitrary; the rank-independent model, where w satisfies the self-conjugacy condition $w(1 - p) = 1 - w(p)$ for all $p \in [0, 1]$; and expected utility, where w is the identity function. Each characterization result is novel and derived from a set of simple axioms, including one key new axiom in each case. We also discuss the challenges of identifying such models when data are limited to simple prospects, how these models extend to general binary prospects, and the implications for empirical model testing.

Keywords: biseparable model, rank-dependence, expected utility, certainty equivalents

JEL codes: D81, D90

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1 Introduction

Let $(x, y; p)$ denote a risky prospect that pays x dollars with probability p, and y dollars with probability $1 - p$, and let $F(x, y; p)$ be its certainty equivalent (CE), i.e. the sum of money for which, in a choice between the money and the prospect, the decision maker is indifferent between the two. In this article, we are interested in individual preferences that lead to the following biseparable model of the certainty equivalent

$$F(x, y; p) = u^{-1}(w(p)u(x) + (1 - w(p))u(y)) \quad \text{for} \quad x, y \in X,\ x \geq y,\ p \in [0, 1], \tag{1}$$

where X is a real interval, $u : X \to \mathbb{R}$ is a strictly increasing and continuous function, called the utility function, and $w : [0, 1] \to [0, 1]$ is a strictly increasing function satisfying $w(0) = 0$ and $w(1) = 1$, called a probability weighting function.

Note that the function w assigns a weight to the probability associated with the higher of the two possible payoffs. Therefore, the weight depends not only on the numerical value of the probability, but also on the rank of the payoff with which it is associated. For this reason, equation (1) is sometimes referred to as a binary rank-dependent model.

A notable special case arises when the weighting function w is independent of the payoff rank. This occurs when w is the identity function, in which case the model reduces to the certainty equivalent (CE) of the binary expected utility (EU) framework. More generally, rank-independence holds if the weighting function w is *self-conjugate*, i.e. satisfies the following condition:

$$w(1 - p) = 1 - w(p) \quad \text{for} \quad p \in [0, 1].$$

In this paper, we formally characterize equation (1), along with its rank-independent and binary expected utility (EU) special cases, over the full domain of binary prospects. In addition, we provide separate characterizations of these models on a restricted domain of simple prospects—that is, binary prospects where one of the two payoffs is held fixed. Specifically, for a fixed payoff $y_0 \in X$, let (x, p) denote a binary prospect yielding x with probability p and y_0 with probability $1 - p$. On this restricted domain, model (1) simplifies to:

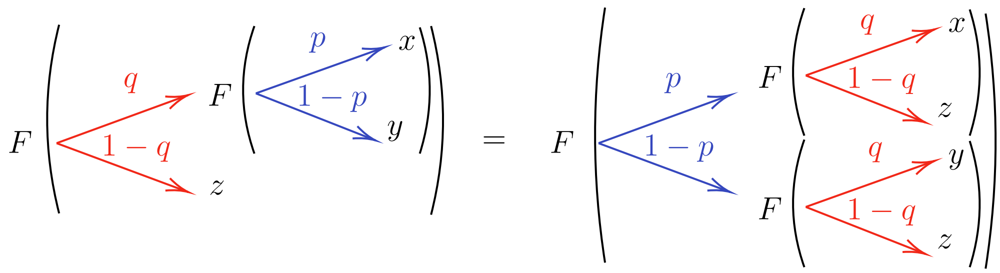
$$F(x, p) = \begin{cases} u^{-1}(w_-(p)u(x)) & \text{for} \quad x < y_0,\ p \in [0, 1], \\ u^{-1}(w_+(p)u(x)) & \text{for} \quad x \geq y_0,\ p \in [0, 1], \end{cases} \tag{2}$$

where u is a utility function satisfying $u(y_0) = 0$ and w_-, w_+ are probability weighting functions.[1]

[1] To see that (1) is an extension (2), suppose that (1) holds for some utility function v and a continuous probability weighting function w. Fix $y_0 \in X$ and define $u(x) := v(x) - v(y_0)$, $w_+(p) = w(p)$ and $w_-(p) = 1 - w(1 - p)$. Then we obtain (2).

Each of our characterizations is novel and derived from a set of simple axioms, including a key novel axiom specific to each case. The general rank-dependent model (1) is obtained from the following *distributivity* axiom:

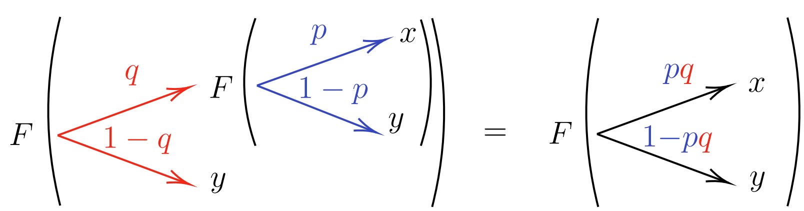
(Dist)
$x \geqslant y \geqslant z$



where $p, q \in (0,1)$. This axiom is based on the idea that the sequential evaluation of independent risks should not depend on the order in which the risks are considered. Suppose that $1-q$ represents the political risk of war, and $1-p$ represents the economic risk of low demand, with the assumption that these risks are independent (i.e., the probability of war is the same under both high and low demand). Further, suppose that profits do not depend on the order in which the risks are resolved, and are defined as follows: z in the case of war, y in the case of peace and low demand, and x in the case of peace and high demand, with $x \geq y \geq z$. The right-hand side of the **(Dist)** axiom corresponds to the process of folding back the decision tree in which the political risk is resolved first, followed by the economic risk. The left-hand side represents the same process with the order of risks reversed. Therefore, the equality of the two sides demonstrates that the order of independent risks in a sequential evaluation process does not affect the outcome.

The ordering of payoffs $x \geq y \geq z$ in **(Dist)** is essential. If the same condition is assumed without this order restriction—denoted as ($\overline{\textbf{Dist}}$)—the resulting model is rank-independent; that is, (1) holds with a self-conjugate weighting function w. The CE of the binary EU model, where w is the identity function, follows from an even stronger condition—the following *reduction* axiom:

(Red2)



This axiom differs from **(Dist)** in several respects. First, it involves only two payoffs instead of three. Second, **(Red2)** is stronger because unlike in the investor story above, the two risks it refers to are not required to be independent. If the risks are independent, then **(Red2)** ensures that the order in which they are evaluated does not affect the outcome.

Table 1: Summary of the representations and the key axioms used to derive them

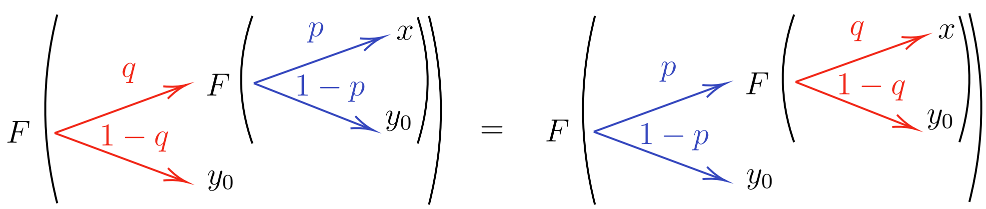
prospects	rank-dependent general w	rank-independent self-conjugate w	EU $w = \text{id}$
simple	**(Perm)**	x	**(Red)**
binary	**(Dist)**	($\overline{\textbf{Dist}}$)	**(Red2)**

More importantly, the axiom also holds when the risks are arbitrarily correlated; in this case, p represents a *conditional* probability—specifically, the probability of high demand given peace in our example. The axiom states that sequential evaluation is equivalent to a one-step, simultaneous evaluation, as shown on the right-hand side of the axiom, where the decision tree is "folded back" in a single step. Finally, **(Red2)** differs from **(Dist)** in that it imposes no ordering restriction on the payoffs.

The corresponding representations for simple prospects are derived from weaker axioms. Specifically, the EU representation for simple prospects—that is, model (2) with w_+ and w_- as identity functions—is obtained from a weakened version of **(Red2)**, where the arbitrary payoff y is replaced by a fixed payoff y_0. This weaker axiom is denoted by **(Red)**. In fact, **(Red2)** can be viewed as a family of **(Red)** axioms, each applied for a different fixed payoff y_0.

The more general representation (2) with arbitrary probability weighting functions w_+ and w_- is derived from the following *permutability* axiom:

(Perm)



(Perm) is similar to **(Dist)**, but simpler. It also involves two binary independent[2] risks with probabilities p and q, which are priced sequentially, and the order in which they are priced does not matter. The difference is that instead of three ordered payoffs, we now have only two payoffs, one of which is fixed. This makes **(Perm)** significantly weaker. Table 1 summarizes all characterizations of this paper together with the key axioms used to derive them.

[2]To see that the risks are now independent—unlike in **(Red2)** or **(Red)**—note that the probabilities p and q appear in both the first and second branches of the decision tree. This implies that the conditional and unconditional probabilities of high demand (or peace) must be equal. In other words, the two risks—political and economic—are uncorrelated.

1.1 A General Class of Preferences

Model (1) is not only of interest in its own right. Drawing on the idea from Ghirardato and Marinacci (2001), who introduced biseparable preferences in the context of uncertainty, we interpret this model as characterizing all risky choice preferences over multi-outcome prospects that yield certainty equivalents (CEs) consistent with (1) when restricted to binary prospects. The class of preferences that generate CEs of the form (1) is quite broad. It includes several popular non-expected utility models that, when applied to binary prospects, coincide with the rank-dependent utility framework.[3] According to (Wakker, 2010, Observation 7.11. 1) these include, for example, the rank-dependent utility model (Quiggin, 1982; Chew and Epstein, 1989), the RAM and TAX model (Birnbaum, 2008), disappointment aversion theory (Gul, 1991), the original prospect theory restricted to gains or losses (Kahneman and Tversky, 1979) or the prospective reference theory (Viscusi, 1989). However, (1) is also compatible with well-defined intransitive preferences that are not represented by a utility function. These include, among others, preferences that, on the set of binary prospects, coincide with the range utility theory (Baucells et al., 2024) or range-dependent utility (Kontek and Lewandowski, 2018). In particular, as shown in Section 3.3, such preferences can incorporate the preference reversal phenomenon (Grether and Plott, 1979). Focusing on the biseparable model (1), we investigate the "common denominator" for all these models.

1.2 Axiomatization on Restricted Domains

Representations for binary prospects are typically derived from representations on a more general domain. For example, Köbberling and Wakker (2003) discusses ways to obtain binary representations as a special case of the general rank-dependent utility model, using trade-off techniques. This 'general to specific' approach has some potential drawbacks. Specifically, axioms that can be stated for any finite number of payoffs sometimes cannot be stated for a fixed number of payoffs. For instance, the well-known characterizations of the quasi-arithmetic mean[4] (Nagumo, 1930; Kolmogorov, 1930; de Finetti, 1931), are based on axioms requiring that means be defined for an arbitrary number of payoffs. The replacement axiom of Kolmogorov (1930) written for two payoffs reduces to a tautology. Similarly, the quasilinearity axiom of de Finetti (1931) uses the idea of a mixture of two probability distributions. While a mixture of two probability distributions with finite support also has finite support,

[3]Luce and Narens (1985) explore the concepts of m-point homogeneity and n-point uniqueness for general scales and find that rank-dependent utility is the most general interval scale for two states of nature. See also Sokolov (2011).

[4]A (weighted) quasi-arithmetic mean with a strictly increasing generator is formally equivalent to a CE of a lottery under the expected utility theory.

mixing two binary distributions does not have to be binary. In Section 3.2 we show that stating the axiom of quasilinearity in the domain of binary prospects is nontrivial. Moreover, we show that even such a binary version of this axiom is still stronger than our reduction axiom, which is used to characterize the certainty equivalent of a binary prospect under the expected utility model.

We build on the approach proposed by Aczél (1947). Instead of means defined over an arbitrary number of payoffs, Aczél was interested in characterizing the mean over a fixed number of payoffs. He proposed the axiom of bisymmetry to characterize the bivariate quasilinear mean.[5] Following this approach, we design axioms that are minimal in the sense that they are tailored to the domain of prospects in which the model applies. In this way, we provide a tool for precisely addressing the problem of extending a given model from a restricted domain to a wider one. For example, we can determine to what extent we need to strengthen the axioms in order to extend a biseparable model from the domain of simple prospects to binary prospects. We address this problem in the discussion that follows our main results in Sections 2.1 and 2.2.

There are works that follow the approach of Aczél (1947) in characterizing certainty equivalents of the form (1) or the binary rank-dependent utility model (Luce and Narens, 1985; Pfanzagl, 1959; Miyamoto, 1988; Luce, 1991; Luce and Fishburn, 1991). These papers use bisymmetry-like conditions (Köbberling and Wakker, 2003, Sections 5.2 and 7), which are based on the following bisymmetry equation written for certainty equivalents:

$$F(F(x,y;p),F(z,t;p);q) = F(F(x,z;q),F(y,t;q);p). \tag{3}$$

The bisymmetry equation (3) shares some basic intuition with **(Dist)** or **(Perm)**, namely that in sequential risk evaluation by backward folding and replacing each chance node by certainty equivalents, the order of evaluation does not matter.

However, our characterizations stand out from those based on bisymmetry in the following ways. First, our approach is *systematic*. We offer characterizations of the biseparable model for both simple and binary prospects; for the general probability weighting function as well as for the case of expected utility and rank independence. Second, we offer a *unified* approach. Each of our characterizations has the same structure based on three conditions: a natural reflexivity axiom, a regularity (continuity and monotonicity) axiom common to all axiomatizations, and one new key axiom specific to the given model. With such a simple and unified structure, it is possible to compare models with each other. For example, comparing axioms (**Perm**) and (**Dist**) tells us how much the condition needs to be strengthened in

[5]For a given real interval X and a mapping $M : X^2 \to X$, bisymmetry holds if $M(M(x,y),M(z,t)) = M(M(x,z),M(y,t))$ is true for all $(x,y,z,t) \in X^4$.

order to extend the biseparable model for simple prospects to the set of binary prospects. Third, our *key axioms are minimal*, i.e. tailored and adapted to the domain over which the model is defined. This makes them less complex and easier to test than others. For example, (**Perm**) gives a biseparable model on the set of simple prospects. In order to test it, for a given value of each of its 3 variables (x, p, q) one needs to elicit the certainty equivalents of 4 prospects each with payoffs x, y_0, where y_0 is a fixed payoff. To extend the biseparable model to the set of binary prospects, (**Dist**) requires two additional variables (y, z) and *one* additional CE elicitation. In comparison, the bisymmetry equation (3) includes three additional variables (y, z, t) and *two* additional CE elicitations.

1.3 Model Identification vs. Model Validation: Implications and Distinctions

A common practice in descriptive and prescriptive approaches to decision support is to train a model on a subset of data and then use it to make predictions on data outside of this subset. In the case of a preference model, the data are provided by participants in preference elicitation experiments. The most popular models can be identified on a small subset of their domain. Therefore, to reduce the cognitive load on subjects and minimize noise, experimental designers most often choose low-complexity task sets. Often, model parameters are identified using certainty equivalents rather than choice data, because the former provide point– rather than interval–estimates. For example, Tversky and Kahneman (1992) use CEs of binary prospects, mostly with a common payoff (simple prospects), to identify parameters of cumulative prospect theory (see also Gonzalez and Wu, 1999). The uniqueness parts of our characterization theorems provide an answer to the question of what data are sufficient to identify model parameters. For example, a rank-dependent utility model for finite-support prospects can be fully identified using the certainty equivalents of binary prospects. However, data on only simple prospects, i.e. binary prospects with a fixed payoff, are insufficient (see Theorem 1a and 1b). These results can help to better design experiments. In Section 3.1 we discuss the implications of our results for model identification.

A similar technique of using low-complexity tasks is used in the normative application of Expected Utility theory. The decision maker is asked to make a series of simple choices (e.g., determining the CEs of binary prospects) in order to identify a utility function that is ultimately used to predict more complex choices, e.g., choosing between prospects with multiple payoffs (Gilboa, 2009, p. 87, see also Luce and Raiffa, 1957, Section 2.8). This approach, in which data from low-complexity tasks are used as input to a model to infer more complex choices, is reasonable if we know that the model is true in the larger domain.

This is the case for model identification. However, to validate the model, we need data drawn from the entire domain. For example, we cannot use data solely for binary prospects to validate a rank-dependent utility model for prospects with multiple payoffs. These data confirm the quality of the model only in the set of binary prospects. Some authors seem to forget this, judging the quality of a model on its entire domain by the quality of the model fit on a subdomain. Since the vast majority of experimental data on choice under risk involve binary prospect data, we can make confident judgments about the quality of the model only for binary prospects. There is no similar consensus for models whose domain includes prospects with more than two payoffs, because there is much less data on such prospects and the existing data are not sufficiently conclusive. This further justifies our interest in the biseparable model.

In the next section, we begin by introducing the model, followed by the presentation of all five characterization results along with a discussion of their key axioms. The results are organized as follows: we start with the representation (2) for simple prospects. For this case, we distinguish between two scenarios depending on whether the fixed outcome y_0 lies at the boundary of the payoff space X or in its interior, each allowing for unrestricted probability weighting functions w_+ and w_-. We then derive the special case corresponding to expected utility. After that, we turn to the representation (1) for binary prospects, beginning with the most general rank-dependent specification, and then proceeding to the rank-independent case, concluding with the binary expected utility model.

2 The Main Characterization Results

A binary prospect is a probability measure on a real interval X with a support consisting of at most two elements. Those elements are called payoffs of a prospect. We let $(x, y; p)$ denote a binary prospect that pays x and y, with probabilities p and $1 - p$, respectively. Note that

$$(x, y; p) = (y, x; 1 - p) \quad \text{for} \quad x, y \in X,\ p \in [0, 1]. \tag{4}$$

If $x = y$ or $p \in \{0, 1\}$ then the prospect $(x, y; p)$ is called degenerate. Any such prospect is identified with its payoff. Consequently, $(x, x; p) = x$ holds for all $x \in X$, $p \in [0, 1]$, and $(x, y; 1) = x$, $(x, y; 0) = y$ hold for all $x, y \in X$. Let $\Delta(X)$ denote the set of all binary prospects. We will also consider families of simple prospects, i.e. binary prospects in which one payoff is fixed. Given $y_0 \in X$, we denote by $\Delta_{y_0}(X)$ the set of simple prospects of the form $(x, y_0; p)$. Whenever we consider simple prospects, we assume that y_0 is known and constant and we write simple prospects as (x, p).

The primitive element of our model is the certainty equivalent of a prospect in the biseparable model, i.e. a functional $F : \Delta(X) \to X$ having the representation (1) for $(x, y; p) \in \Delta(X)$. From now on, a utility function is a strictly increasing and continuous function $u : X \to \mathbb{R}$ and a probability weighting function is a strictly increasing function $w : [0, 1] \to [0, 1]$ satisfying $w(0) = 0$ and $w(1) = 1$. Our basic axiom common to all axiomatizations is *reflexivity*:

(Ref) *$F(x) = x$ for all degenerate prospects $x \in \Delta(X)$.*

In all our results, we also impose the following regularity condition.

(CM) *F is strictly increasing and continuous in the probability of the higher payoff.*

2.1 Characterizations for simple prospects

We first consider representation (1) for simple prospects $\Delta_{y_0}(X)$, given $y_0 \in X$. The key axiom is permutability,[6] which has been graphically presented in Section 1.

(Perm) *$F(F(x, p); q) = F(F(x, q); p)$ for $x \in X$ and $p, q \in (0, 1)$.*

We formulate characterization results based on this axiom for the case when y_0 is the endpoint of X and separately for the case where y_0 belongs to the interior of X. Proofs of all theorems in this paper are provided in the Appendix.

Theorem 1a (**simple only gains/only losses**) Assume y_0 is an endpoint of X. A function $F : \Delta_{y_0}(X) \to X$ satisfies **(Ref)**, **(CM)**, and **(Perm)** if and only if there exist a continuous probability weighting function w and a utility function u satisfying $u(y_0) = 0$ such that

$$F(x, p) = u^{-1}(w(p)u(x)) \quad \text{for} \quad x \in X,\ p \in [0, 1]. \tag{5}$$

Furthermore, (5) is satisfied with w replaced by another probability weighting function $\tilde{w}$, and u replaced by another utility function $\tilde{u}$ satisfying $\tilde{u}(y_0) = 0$ if and only if there exist $\alpha, r > 0$ such that

$$\tilde{w}(p) = w(p)^r \qquad \text{for} \quad p \in [0, 1], \tag{6}$$

$$|\tilde{u}(x)| = \alpha |u(x)|^r \quad \text{for} \quad x \in X. \tag{7}$$

[6]The name derives from the fact that the axiom requires the one-parameter set of mappings $y = F(x, p)$ of X in X to be permutable.

Theorem 1b (**simple gains and losses**) Let y_0 belong to the interior of X. A function $F : \Delta_{y_0}(X) \to X$ satisfies **(Ref)**, **(CM)**, and **(Perm)** if and only if there exist continuous probability weighting functions w_-, w_+, and a utility function u satisfying $u(y_0) = 0$ such that (2) holds, i.e.

$$F(x,p) = \begin{cases} u^{-1}(w_-(p)u(x)) & \text{for} \quad x < y_0,\ p \in [0,1], \\ u^{-1}(w_+(p)u(x)) & \text{for} \quad x \geq y_0,\ p \in [0,1]. \end{cases}$$

Furthermore, (2) is satisfied with w_-, w_+ replaced by another pair of probability weighting functions $\tilde{w}_-, \tilde{w}_+$, and u replaced by another utility function $\tilde{u}$ satisfying $\tilde{u}(y_0) = 0$ if and only if there exist $\alpha, \beta, r_-, r_+ > 0$ such that

$$\tilde{w}_-(p) = w_-(p)^{r_-} \quad \text{for} \quad p \in [0,1], \tag{8}$$

$$\tilde{w}_+(p) = w_+(p)^{r_+} \quad \text{for} \quad p \in [0,1], \tag{9}$$

and

$$\tilde{u}(x) = \begin{cases} -\alpha(-u(x))^{r_-} & \text{for} \quad x < y_0, \\ \beta u(x)^{r_+} & \text{for} \quad x \geq y_0. \end{cases} \tag{10}$$

We now characterize the special case of (5) or (2) where the probability weighting function is the identity function. The key axiom in this case is reduction.[7]

(Red) $F(F(x,p),q) = F(x,pq)$ *for* $x \in X,\ p,q \in (0,1)$.

Theorem 2 (**simple EU**) *Let* $y_0 \in X$*. A function* $F : \Delta_{y_0}(X) \to X$ *satisfies* **(Ref)**, **(CM)** *and* **(Red)** *if and only if there exists a utility function* u *satisfying* $u(y_0) = 0$ *and:*

$$F(x,p) = u^{-1}(pu(x)) \quad \text{for} \quad x \in X,\ p \in [0,1]. \tag{11}$$

Furthermore, (11) *is satisfied with* u *replaced by another utility function* $\tilde{u}$ *satisfying* $\tilde{u}(y_0) = 0$ *if and only if:*

- *in the case where* y_0 *is the endpoint of* X*, there exists* $\alpha > 0$ *such that*

$$\tilde{u}(x) = \alpha u(x) \quad \text{for} \quad x \in X; \tag{12}$$

[7] In the field of functional equations the term translation equation is used. Moszner (1995) gives a survey of results on its solutions.

- *in the case where y_0 is the interior point of X, there exist $\alpha, \beta > 0$ such that*

$$\tilde{u}(x) = \begin{cases} \alpha u(x) & \text{for } x < y_0, \\ \beta u(x) & \text{for } x \geq y_0. \end{cases} \tag{13}$$

(Red) is stronger than **(Perm)**. Indeed, **(Red)** implies **(Perm)**, which can be seen by applying **(Red)** on both sides of **(Perm)**, but the opposite is not true.[8]

2.2 Characterization results for binary prospects

We now characterize the model for all binary prospects. The key axiom we apply in this case is distributivity.

(Dist) $F(F(x, y; p), z; q) = F(F(x, z; q), F(y, z; q); p)$ *for* $x \geq y \geq z,\ p, q \in (0, 1)$.

Theorem 3 (**binary**) *A function $F : \Delta(X) \to X$ satisfies the axioms* **(Ref)**, **(CM)**, *and* **(Dist)** *if and only if there exist a utility function u and a continuous probability weighting function w, such that* (1) *holds, i.e.*

$$F(x, y; p) = u^{-1}(w(p)u(x) + (1 - w(p))u(y)) \quad \text{for} \quad x \geq y,\ p \in [0, 1].$$

Furthermore, (1) *is satisfied with u replaced by another utility function $\tilde{u}$, and w replaced by another probability weighting function $\tilde{w}$ if only if $\tilde{w} = w$ and there exist $\alpha, \beta \in \mathbb{R}$, with $\alpha > 0$, such that $\tilde{u}(x) = \alpha u(x) + \beta$ holds for all $x \in X$.*

Note that the model of Theorem 3 is an extension of the models of Theorem 1a and 1b. The representation in (1) gives the form of $F(x, y; p)$ for $x \geq y$, i.e. for a given payoff rank. In other cases we use (4) to get

$$F(x, y; p) = u^{-1}(\bar{w}(p)u(x) + (1 - \bar{w}(p))u(y)) \quad \text{for } x < y,\ p \in [0, 1]. \tag{14}$$

where $\bar{w} : [0, 1] \to [0, 1]$ is a function defined by

$$\bar{w}(p) = 1 - w(1 - p) \quad \text{for} \quad p \in [0, 1].$$

The formulas in (1) and (14) differ in general because the weight assigned to a given payoff may depend not only on the probability of that payoff occurring, but also on the rank

[8] For example, the representation $F(x, p) = xw(p)$, for $x \in X$, $p \in [0, 1]$, satisfies **(Perm)** for any probability weighting function w satisfying the conditions of Theorem 1b. On the other hand, this representation satisfies **(Red)** if and only if $w(pq) = w(p)w(q)$ for all $p, q \in [0, 1]$ which is true if and only if $w(p) = p^\alpha$, $p \in [0, 1]$ for some $\alpha > 0$.

of the payoff. For example the weight assigned to x equals $w(p)$ if $x \geq y$ and $\bar{w}(p)$ if $x < y$. This is reflected in the **(Dist)** axiom, which holds for ordered payoffs $x \geq y \geq z$. We thus say that the model is *rank-dependent*. Note, however, that the formulas for $x \geq y$ and $x < y$, given by (1) and (14), would coincide if w was *self-conjugate*, i.e. it satisfied the following condition

$$\bar{w} = w. \tag{15}$$

We call a biseparable model that meets this condition *rank-independent*. It can be obtained by strengthening the **(Dist)** axiom to hold for any payoffs x, y, z and not just for ordered payoffs.

($\overline{\textbf{Dist}}$) $F(F(x,y;p),z;q) = F(F(x,z;q),F(y,z;q);p)$ *for* $x,y,z \in X,\ p,q \in (0,1)$.

We can derive a rank-independent model as a corollary to Theorem 3.

Corollary 1 (**rank-independent**) *A function $F : \Delta(X) \to X$ satisfies the axioms* **(Ref)**, **(CM)**, *and* **($\overline{\textbf{Dist}}$)** *if and only if there exist a utility function u and a self-conjugate continuous probability weighting function w, such that*

$$F(x,y;p) = u^{-1}(w(p)u(x) + (1-w(p))u(y)) \quad \textit{for} \quad (x,y;p) \in \Delta(X). \tag{16}$$

Uniqueness is as in Theorem 3.

Specializing the model further, we now consider the case where the probability weighting function w is the identity function. The key axiom is now the natural extension of **(Red)**:

(Red2) $F(F(x,y;p),y;q) = F(x,y;pq)$ *for* $x,y \in X,\ p,q \in (0,1)$.

Theorem 4 (**binary EU**) *A function $F : \Delta(X) \to X$ satisfies* **(Ref)**, **(CM)** *and* **(Red2)** *if and only if there exists a utility function u such that*

$$F(x,y;p) = u^{-1}(pu(x) + (1-p)u(y)) \quad \textit{for} \quad (x,y;p) \in \Delta(X). \tag{17}$$

Furthermore, (17) *is satisfied with u replaced by another utility function $\tilde{u}$ if and only if there are $\alpha, \beta \in \mathbb{R}$, with $\alpha > 0$, such that*

$$\tilde{u}(x) = \alpha u(x) + \beta \quad \textit{for} \quad x \in X. \tag{18}$$

It is instructive to compare the key axioms of the models for binary prospects with those for simple prospects. Note that while the second prospect payoff in **(Red)** is fixed at y_0, **(Red2)**, in which the corresponding payoff y is arbitrary, is equivalent to the system of

(Red) for each y_0. This stronger axiom is sufficient to extend the simple EU to the binary EU model. One might think that an analogous generalization of **(Perm)**, the key axiom of the biseparable model for simple prospects, is sufficient to extend this model from simple to binary prospects. However, this is not the case. Indeed, consider the following natural extension of **(Perm)**:

(Perm2) $F(F(x, y; p), y; q) = F(F(x, y; q), y; p)$ *for* $x \geq y,\ p, q \in (0, 1)$.

For some utility function $\phi : X \to \mathbb{R}$, the model

$$F(x, y; p) = y + \phi^{-1}(p\phi(x - y)), \quad x \geq y,\ p \in [0, 1] \tag{19}$$

satisfies **(Perm2)** but is not biseparable in general. To obtain a biseparable model, one must use the stronger **(Dist)** axiom, of which **(Perm2)** is a special case obtained by setting $z = y$ in the former. It is obvious that **(Perm)** is weaker than both of the above axioms and can be obtained from **(Dist)** by setting $z = y = y_0$.

Even though **(Red)** and **(Perm)** apply on the set of simple, while **(Red2)**, **(Perm2)** and **(Dist)** on the set of binary prospects, it is instructive to compare all of them on the larger set of all binary prospects $\Delta(X)$. The logical relationships between the axioms on this set are depicted below:

$$\begin{array}{ccccc} (\textbf{Dist}) & \Longrightarrow & (\textbf{Perm2}) & \Longleftarrow & (\textbf{Red2}) \\ & & \Downarrow & & \Downarrow \\ & & (\textbf{Perm}) & \Longleftarrow & (\textbf{Red}) \end{array}$$

We finish this section with the observation that the regularity axiom **(CM)** assumes continuity and monotonicity only with respect to the probability of the higher payoff. Yet, the representations of Theorems 1a–4 imply stronger versions of continuity and monotonicity. In fact, F in all these representations is continuous in each of its variables and monotonic with respect to first-order stochastic dominance (FOSD).[9]

3 Discussion

We discuss implications of our results for simple prospects for model identification. Next, we illustrate the advantage of our axioms for binary prospects as compared to axioms deduced

[9] For any pair of prospects $(x, y; p)$ and $(x', y', ; p')$, where $x > y$, $x' > y'$ and $p, p' \in (0, 1)$ we say that $(x, y; p)$ dominates $(x', y'; p')$ if the following inequalities hold: $x \geq x'$, $y \geq y'$ and $p \geq p'$, and at least one of these inequalities is strict. We say that F is monotonic with respect to FOSD if $F(x, y; p) > F(x', y'; p')$ holds whenever $(x, y; p)$ dominates $(x', y'; p')$. Monotonicity with respect to FOSD thus holds if F is strictly increasing in payoffs and in the probability of the higher payoff.

as a special case of the model for general prospects. Finally, we show that the biseparable model allows for preference reversals while the binary rank-dependent utility model does not.

3.1 On the Identifiability of Models for Simple Prospects

The uniqueness part of Theorem 1b shows that model (2) is only partially identified on the set of simple prospects. One can think of y_0 in simple prospects as a reference point, relative to which other payoffs are evaluated. Any payoff x above y_0 is treated as gain and any payoff below y_0 as loss. The first straightforward implication of Theorem 1b is that the ratio of utilities for gains and losses (i.e loss aversion) is unidentified and hence the utility scale consists of two separate scales, one for gains and one for losses. Simple prospects are insufficient to identify loss aversion, because a simple prospect is either a gain prospect or a loss prospect, but never a mixed prospect.

Second, Theorem 1b implies that by taking the positive powers of the utility function and the probability weighting function separately for the gain and loss parts, we obtain an equivalent representation. This has important consequences in modeling individual attitudes towards risk. A typical shape for a probability weighting function is an inverted S-shaped function (Wakker, 2010, p.204), in which (winning) probabilities below a certain threshold are overweighted and probabilities above the threshold are underweighted. The location of the threshold, which is the interior fixed point of the probability weighting function, is an important element in modeling individual attitudes towards risk. Theorem 1b implies that for simple prospects, which many empirical studies use only (Preston and Baratta, 1948) or mostly (Tversky and Kahneman, 1992; Gonzalez and Wu, 1999), the threshold cannot be identified.

To illustrate this, let's fix $y_0 \in X$ and assume that F is of the form (2) for some utility function u such that $u(y_0) = 0$ and probability weighting functions w_+, w_-. Consider any point $p_0 \in (0, 1)$ that is not a fixed point of w_+ (the same argument applies also separately to w_- or to w_+, w_- jointly). Since $w_+(p_0) \in (0, 1)$, for any such point one can find $r > 0$ such that $w_+(p_0) = p_0^r$. Let's define a new pair of functions $\tilde{w}_+, \tilde{u}$ by $\tilde{w}_+(p) = w_+(p)^{1/r}$, $\tilde{u}(x) = u(x)^{1/r}$ for $x \geq y_0$ and $\tilde{u}(x) = u(x)$ for $x < y_0$ which, according to Theorem 1b, generate an equivalent representation. Note that although p_0 is not a fixed point of w_+, it is a fixed point of $\tilde{w}_+$:

$$\tilde{w}_+(p_0) = w_+(p_0)^{1/r} = (p_0^r)^{1/r} = p_0.$$

In this way, an equivalent representation can be constructed in which the probability weighting function has a fixed point at any desired point in $(0, 1)$. This observation is visually illustrated in Figure 1, which shows several equivalent pairs of utility functions and probability weighting functions, each of the latter with a different interior fixed point.

(a) $\tilde{u}(x) = u(x)^r$, $x \geq y_0 = 0$ (b) $\tilde{w}_+(p) = w_+(p)^r$, $r > 0$

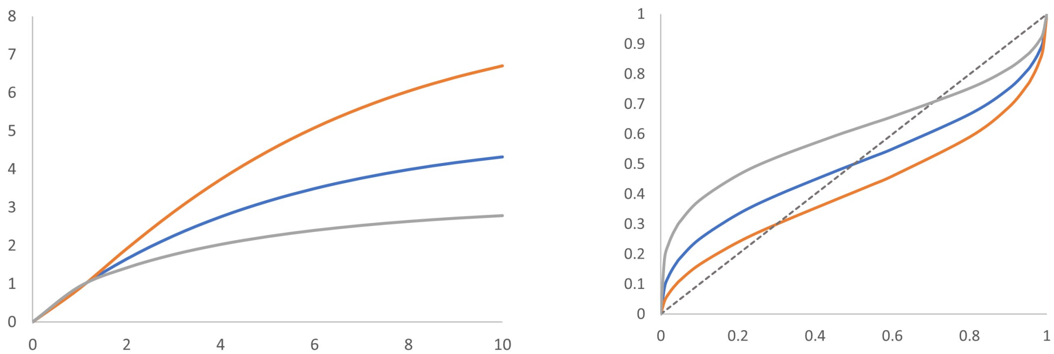


Figure 1: Three different pairs of functions $(\tilde{u}, \tilde{w}_+)$, each shown in a different color, produce the same certainty equivalent $F(x, p)$.

Moreover, if u is of the following form:[10]

$$u(y) = \begin{cases} y^{\alpha_+} & \text{for} \quad y \geq 0, \\ -\lambda(-y)^{\alpha_-} & \text{for} \quad y < 0, \end{cases} \tag{20}$$

for some $\alpha_+, \alpha_- > 0$ and $\lambda > 0$, then the uniqueness part of Theorem 1b allows us to obtain an equivalent representation of $F(y, p)$ in which the utility function is piecewise linear with the loss aversion parameter $\tilde{\lambda} = \lambda^{1/\alpha_-}$, i.e. it is of the following form:

$$\tilde{u}(y) = \begin{cases} y & \text{for} \quad y \geq 0, \\ \tilde{\lambda} y & \text{for} \quad y < 0. \end{cases} \tag{21}$$

Indeed, it is enough to define $\tilde{w}_+(p) = w_+(p)^{1/\alpha_+}$, $\tilde{w}_-(p) = w_-(p)^{1/\alpha_-}$ for $p \in [0, 1]$ and

$$\tilde{u}(y) = \begin{cases} u(y)^{1/\alpha_+} & \text{for} \quad y \geq 0, \\ -(-u(y))^{1/\alpha_-} & \text{for} \quad y < 0. \end{cases}$$

We can specialize the above observation even further if we additionally assume that w_+, w_- are Prelec (1998) functions, i.e.

$$w_i(p) = \begin{cases} 0 & \text{for} \quad p = 0, \\ (\exp(-(-\ln p)^{\gamma_i}))^{\beta_i} & \text{for} \quad p \in (0, 1], \end{cases} \tag{22}$$

[10]This a popular form in reference dependent models, where loss aversion plays an important role. Loss aversion occurs when $\lambda > 1$. In what follows, we will fix y_0 and normalize payoffs relative to it. So instead of the original payoff $x \in X$ we will use the normalized payoff $y := x - y_0$.

with some $\beta_i > 0$ and $\gamma_i \in (0,1)$ for $i \in \{+,-\}$. Then, the following two representations of $F(y,p)$ given by (2) are equivalent:

- $w_i(p) = (\exp(-(-\ln p)^{\gamma_i}))^{\beta_i/\alpha_i}$ for $p \in (0,1]$, $i \in \{+,-\}$, and u is of the form (21).
- $w_i(p) = \exp(-(-\ln p)^{\gamma_i})$ for $p \in (0,1]$ and u is of the following form:

$$u(y) = \begin{cases} y^{\alpha_+/\beta_+} & \text{for} \quad y \geq 0, \\ -\lambda(-y)^{\alpha_-/\beta_-} & \text{for} \quad y < 0. \end{cases}$$

It means that we can either set the curvature parameters of the utility function α_+, α_- equal to 1, thus making it piecewise linear, or we can set the probability weighting function parameters β_+, β_- equal to 1, thus fixing their unique interior fixed point[11] at $p = \exp(-1)$. Hence, the necessary and sufficient condition for w_-, w_+ to be 1-parameter Prelec functions with a fixed point at $p_0 = \exp(-1)$ and u to be piecewise-linear, is that $\alpha_+ = \beta_+$ and $\alpha_- = \beta_-$.

3.2 Illustrating the Domain-Specificity of the Axioms

The important property of all our axioms is that they are defined one the same domain as the representation they yield. This is not true in many other axiomatizations. For example (Ghirardato and Marinacci, 2001, Theorem 3) characterize a biseparable representation under uncertainty, in which the domain of their key axiom (Weak Certainty Independence) goes beyond the set of binary acts.

We illustrate this property by comparing our reduction axiom for binary prospects with the quasilinearity axiom of de Finetti (Hardy et al., 1934, p. 157-163), used as the key axiom to characterize the quasilinear mean. The quasilinearity axiom is stated for mean values (denoted by F) of finite distribution functions $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ on a bounded real interval $[a,b]$ and their probability mixtures:[12]

(QL) *If* $F(\mathbf{X}) = F(\mathbf{Y})$, *then* $F(\mathbf{X}, \mathbf{Z}; q) = F(\mathbf{Y}, \mathbf{Z}; q)$ *for all* $\mathbf{Z}$ *and* $q \in (0,1)$.

Suppose we want to obtain a version of **(QL)** in the domain of binary prospects. In order to do it, it does not suffice to restrict $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ to be binary. However, applying **(QL)** for $\mathbf{X} = (x, y; p)$, $\mathbf{Y} = (x', y; p')$ and $\mathbf{Z} = y$ where $x, x', y \in X$, $p, p' \in [0,1]$, we obtain

[11] A function of the form (22) with $\beta_i = 1$ is also called a 1-parameter Prelec function. The interior fixed point of the function given by (22) is located at $p_0 = \exp\left(-\left(\beta_i^{\frac{1}{1-\gamma_i}}\right)\right)$. For $\beta_i = 1$ it is thus $p_0 = \exp(-1)$.

[12] Note that this axiom can be viewed as the analogue of the independence (also called substitution) axiom for preferences stated in terms of CEs.

$$\text{If } F(x,y;p) = F(x',y;p'), \text{ then } F(x,y;pq) = F(x',y;p'q) \quad \text{for} \quad q \in (0,1). \tag{23}$$

Hence, in the domain of binary prospects, **(QL)** implies (23). Note that if **(Ref)** is true, then **(Red2)** is equivalent to the restricted version of (23) in which $p' = 1$, i.e.

$$\text{If } F(x,y;p) = x' \text{ then } F(x,y;pq) = F(x',y;q) \quad \text{for} \quad q \in (0,1). \tag{24}$$

The above argument shows that the version of **(QL)** for binary prospects is still stronger, given **(Ref)**, than our domain-specific axiom **(Red2)**. This illustrates that axioms tailored to a given subdomain of prospects (binary or simple prospects) are more efficient and minimal than axioms for a larger domain restricted to hold on a smaller domain.

3.3 Modeling Preference Reversals: The Advantage of Certainty Equivalents

The primitive in our approach is the certainty equivalent. Most often, however, the preference relation is considered primitive. We say that a real-valued function U on a set of prospects Δ represents $\succcurlyeq \subseteq \Delta^2$ if $\mathbf{X} \succcurlyeq \mathbf{Y} \iff U(\mathbf{X}) \geq U(\mathbf{Y})$ for all $\mathbf{X}, \mathbf{Y} \in \Delta$. Note that if $\succcurlyeq$ is transitive and monotonic and a unique CE exists for every prospect, then the Certainty Equivalent functional F represents $\succcurlyeq$. However, there are well-defined nontransitive preferences that are not represented by any function. Such preferences exhibit preference reversal, where one prospect is preferred over the other in direct choice but has a lower certainty equivalent (Lichtenstein and Slovic, 1971; Grether and Plott, 1979; Seidl, 2002). Consider the following preference[13] relation $\succcurlyeq \subset \Delta_0(X) \times \Delta_0(X)$

$$(x,p) \succcurlyeq (y,q) \iff pw^{-1}\left(\tfrac{u(x)}{u(\max(x,y))}\right) \geq qw^{-1}\left(\tfrac{u(y)}{u(\max(x,y))}\right), \tag{25}$$

where $u : \mathbb{R} \to \mathbb{R}$ and $w : [0,1] \to [0,1]$ are strictly increasing and invertible functions satisfying $u(0) = 0$ and $w(0) = 0$, $w(1) = 1$. This model yields CE of the form (5).

$$(F(x,p),1) \sim (x,p) \iff w^{-1}\left(\tfrac{u(F(x,p))}{u(x)}\right) = p \iff F(x,p) = u^{-1}(w(p)u(x)).$$

Thus the CE yields the following order over prospects:

$$F(x,p) \geq F(y,q) \iff \frac{u(x)}{u(y)} \geq \frac{w(q)}{w(p)}. \tag{26}$$

[13]This is a special case of Range Utility Theory (Baucells et al., 2024) for simple prospects in $\Delta_0(X)$.

On the other hand, assuming that $0 < x < y$ and $0 < q < p < 1$, we get

$$(x,p) \succcurlyeq (y,q) \iff pw^{-1}\left(\frac{u(x)}{u(y)}\right) \geq q \iff \frac{u(x)}{u(y)} \geq w\left(\frac{q}{p}\right). \tag{27}$$

Therefore, unless the weighting function w is a power function—which implies $w\left(\frac{q}{p}\right) = \frac{w(q)}{w(p)}$ for all $0 < q < p < 1$—the orderings in (27) and (26) generally differ, and preference reversals can occur. For example, following (Baucells et al., 2024, Section 3.4), consider the prospects $(x,p) = (40, 0.6)$ and $(y,q) = (70, 0.3)$, with the utility function defined as

$$u(x) = \begin{cases} x^{0.8}, & x \geq 0, \\ -2|x|^{0.8}, & x < 0, \end{cases}$$

and the probability weighting function given by

$$w(p) = \frac{p^{0.5}}{p^{0.5} + (1-p)^{0.5}}, \quad p \in [0,1].$$

Then, the certainty equivalents are $F(x,p) = 19$ and $F(y,q) = 22$, yet $w\left(\frac{q}{p}\right) = 0.5 \leq \frac{u(x)}{u(y)} = 0.64$, which implies $(x,p) \succcurlyeq (y,q)$ according to (27). This leads to a preference reversal, as the ranking implied by the certainty equivalents contradicts the ranking given by the direct preference condition.

The above argument shows that, assuming that CEs exist, the class of preferences generating CEs of the form (1) is noticeably more general than the preferences in the binary rank-dependent utility model. This further justifies our focus on the (1) model.

4 Conclusions

In this article, we characterized certainty equivalents of the form (1) for simple and binary prospects. The results help to understand the limitations of the popular method of eliciting preferences for simple or binary prospects and extrapolating the results to more complex prospects, either for descriptive or decision-support purposes. Additionally, our results on the uniqueness of the representations for simple and binary prospects provide us with guidance for testing and identifying models on various datasets. These results may be helpful in designing experiments aimed at eliciting individual attitudes towards risk.

Future research will focus on providing analogous characterization results: a) in the domain of ambiguity/uncertainty in which objective probabilities are unknown to the decision maker, b) for preferences instead of certainty equivalents c) for prospects with more than two payouts. Such characterizations should complement knowledge about how much stronger the conditions should be to extend a given representation for a given domain to a larger domain.

A Proofs

We start with four lemmas that will be used in the proofs of Theorems 1a–4.

Lemma 1 *Let I be a real interval and $y_0 \in I$ be its endpoint. Assume that $u_1, u_2 : I \to \mathbb{R}$ are utility functions with $u_1(y_0) = u_2(y_0) = 0$ and w_1, w_2 are continuous probability weighting functions. Then*

$$u_1^{-1}(w_1(p)u_1(x)) = u_2^{-1}(w_2(p)u_2(x)) \quad \textit{for} \quad x \in I,\ p \in [0,1], \tag{28}$$

if and only if there exist $\alpha, r \in (0, \infty)$ such that

$$w_2(p) \ = w_1(p)^r \qquad \textit{for} \quad p \in [0,1], \tag{29}$$

$$|u_2(x)| = \alpha |u_1(x)|^r \quad \textit{for} \quad x \in I. \tag{30}$$

Proof. Standard computations show that (29) and (30) imply (28). For the converse part, assume that (28) holds. First consider the case when $y_0 = \min I$. Replacing in (28) x by $u_1^{-1}(x)$ and putting

$$f := u_2 \circ u_1^{-1}, \tag{31}$$

we obtain

$$f(w_1(p)x) = w_2(p)f(x) \quad \text{for} \quad x \in u_1(I),\ p \in [0,1].$$

Setting $x = x_0 \in u_1(I) \setminus \{0\}$ gives

$$w_2(p) = \frac{f(w_1(p)x_0)}{f(x_0)} \quad \text{for} \quad p \in [0,1] \tag{32}$$

and so plugging it back yields

$$f(w_1(p)x) = \frac{f(w_1(p)x_0)}{f(x_0)} f(x) \quad \text{for} \quad x \in u_1(I),\ p \in [0,1].$$

Moreover, as w_1 is a continuous probability weighting function, we have $w_1([0,1]) = [0,1]$. Thus we get the following Pexider equation on a restricted domain

$$f(xy) = \frac{f(yx_0)}{f(x_0)} f(x) \quad \text{for} \quad (x,y) \in u_1(I) \times [0,1]. \tag{33}$$

Note that as u_1 is a utility function and $u_1(y_0) = 0$, $u_1(I)$ is a real interval having 0 as its left endpoint. Thus the interior of the domain is an open rectangle contained in $(0, \infty)^2$. Hence, according to (Sobek, 2010, Corollary 2) the solutions of (33) can be uniquely extended to the solutions of the corresponding Pexider equation on $(0, \infty)^2$. So, as f is strictly increasing and continuous with $f(0) = 0$, using the standard results (see for example Theorem 13.1.6 of Kuczma, 2008), we conclude that there exist $\alpha, r \in (0, \infty)$ such that

$$f(x) = \alpha x^r \quad \text{for} \quad x \in u_1(I)$$

and

$$\frac{f(yx_0)}{f(x_0)} = y^r \quad \text{for} \quad y \in [0, 1].$$

Hence, in view of (31) and (32), we obtain (30) and (29), respectively, which completes the proof for the case when $y_0 = \min I$.

We now assume that $y_0 = \max I$. Let $\tilde{I} := \{2y_0 - x : x \in I\}$ and $\tilde{u}_i : \tilde{I} \to \mathbb{R}$ for $i \in \{1, 2\}$ be given by

$$\tilde{u}_i(x) = -u_i(2y_0 - x) \quad \text{for} \quad x \in \tilde{I}. \tag{34}$$

Then $y_0 = \min \tilde{I}$ and, for $i \in \{1, 2\}$, $\tilde{u}_i$ is a utility function with $\tilde{u}_i(y_0) = 0$. Moreover, in view of (28) and (34), we have

$$\tilde{u}_1^{-1}(w_1(p)\tilde{u}_1(x)) = \tilde{u}_2^{-1}(w_2(p)\tilde{u}_2(x)) \quad \text{for} \quad x \in \tilde{I},\ p \in [0, 1].$$

Therefore, applying the already proved part, we conclude that there exist $\alpha, r \in (0, \infty)$ such that (29) holds and $\tilde{u}_2(x) = \alpha \tilde{u}_1(x)^r$ for $x \in \tilde{I}$. Hence, taking (34) into account, we get (30) and the proof is completed. □

Lemma 2 *Let $X = [y_0, a)$ for some $a \in (y_0, \infty)$. Assume that for every $b \in (y_0, a)$ there exist a continuous probability weighting function w_b and a utility function $u_b : [y_0, b] \to \mathbb{R}$ such that $u_b(y_0) = 0$ and*

$$F(x, p) = u_b^{-1}(w_b(p)u_b(x)) \quad \textit{for} \quad x \in [y_0, b],\ p \in [0, 1]. \tag{35}$$

Then $w_z = w_{z'} =: w$ for $z, z' \in (y_0, a)$ and there exists a utility function $u : X \to \mathbb{R}$, with $u(y_0) = 0$, such that (5) holds.

Proof. Let $(a_n : n \in \mathbb{N})$ be a strictly increasing sequence of elements of X such that $\lim_{n\to\infty} a_n = a$. Then, according to (35), for every $n \in \mathbb{N}$, we have

$$u_{a_n}^{-1}(w_{a_n}(p)u_{a_n}(x)) = u_{a_1}^{-1}(w_{a_1}(p)u_{a_1}(x)) \quad \text{for} \quad x \in [y_0, a_1],\ p \in [0, 1].$$

Hence, applying Lemma 1, we obtain that for every $n \in \mathbb{N}$ there exist $c_n, r_n \in (0, \infty)$ such that

$$u_{a_n}(x) = c_n u_{a_1}(x)^{r_n} \quad \text{for} \quad x \in [y_0, a_1],\ n \in \mathbb{N} \tag{36}$$

and

$$w_{a_n}(p) = w_1(p)^{r_n} \quad \text{for} \quad p \in [0, 1],\ n \in \mathbb{N}. \tag{37}$$

From (35) we derive that

$$u_{a_n}^{-1}(w_{a_n}(p)u_{a_n}(a_n)) = u_{a_{n+1}}^{-1}(w_{a_{n+1}}(p)u_{a_{n+1}}(a_{n+1})) \quad \text{for} \quad n \in \mathbb{N},\ p \in [0, 1].$$

Hence, in view of (37), we get

$$u_{a_n}^{-1}(w_1(p)^{r_n}u_{a_n}(a_n)) = u_{a_{n+1}}^{-1}(w_1(p)^{r_{n+1}}u_{a_{n+1}}(a_{n+1})) \quad \text{for} \quad n \in \mathbb{N},\ p \in [0, 1].$$

So, taking $p_n \in (0, 1]$ such that $w_1(p_n) = \left(\frac{u_{a_n}(a_1)}{u_{a_n}(a_n)}\right)^{\frac{1}{r_n}}$, for any $n \in \mathbb{N}$ we obtain

$$\left(\frac{u_{a_n}(a_n)}{u_{a_n}(a_1)}\right)^{\frac{1}{r_n}} = \left(\frac{u_{a_{n+1}}(a_n)}{u_{a_{n+1}}(a_1)}\right)^{\frac{1}{r_{n+1}}}.$$

Thus, in view of (36), we get

$$\left(\frac{u_{a_n}(a_n)}{c_n}\right)^{\frac{1}{r_n}} = \left(\frac{u_{a_{n+1}}(a_n)}{c_{n+1}}\right)^{\frac{1}{r_{n+1}}} \quad \text{for} \quad n \in \mathbb{N}. \tag{38}$$

Define a function $u : X \to \mathbb{R}$ in the following way

$$u(x) = u_{a_1}(x) \quad \text{for} \quad x \in [y_0, a_1], \tag{39}$$

$$u(x) = \left(\frac{u_{a_{n+1}}(x)}{c_{n+1}}\right)^{\frac{1}{r_{n+1}}} \quad \text{for} \quad x \in (a_n, a_{n+1}],\ n \in \mathbb{N}. \tag{40}$$

From (38)-(40) we derive that, for every $n \in \mathbb{N}$, u is continuous on $[a_n, a_{n+1}]$ and so, it is continuous. Moreover u, being strictly increasing on $[a_n, a_{n+1}]$ for $n \in \mathbb{N}$, is strictly increasing. It follows from (39) that $u(y_0) = u_1(y_0) = 0$. Finally, taking $x \in (y_0, a)$ and setting $m := \min\{n \in \mathbb{N} : x \leq a_n\}$, in view of (35)-(37), for any $p \in [0, 1]$, we obtain

$$F(x, p) = u_{b_n}^{-1}(w_{b_n}(p)u_{a_n}(x)) = u_{a_n}^{-1}(w_1(p)^{r_n}c_n u(x)^{r_n}) = u^{-1}(w_1(p)u(x)).$$

Taking **(Ref)** into account, we have also

$$F(y_0,p) = F(y_0) = y_0 = u^{-1}(0) = u^{-1}(w_1(p)u(y_0)).$$

In this way we have proved that (5) holds with $w := w_1$, which completes the proof. $\square$

Lemma 3 *Assume that I is a real interval, $u_1, u_2 : I \to \mathbb{R}$ are utility functions and $\gamma, \theta \in (0,1)$. Then*

$$u_1^{-1}(\gamma u_1(x) + (1-\gamma)u_1(y)) = u_2^{-1}(\theta u_2(x) + (1-\theta)u_2(y)) \quad \textit{for} \quad x, y \in I,\ x \geq y. \tag{41}$$

if and only if $\gamma = \theta$ and there exist $\alpha \in (0,\infty)$ and $\beta \in \mathbb{R}$ such that

$$u_2(x) = \alpha u_1(x) + \beta \quad \textit{for} \quad x \in I. \tag{42}$$

Proof. The 'if' part is standard. We now prove the 'only if' part. Assume that (41) holds and let f be given by (31). Then, as u_1 and u_2 are utility functions, f is strictly increasing and continuous. Furthermore, replacing in (41) x and y by $u_1^{-1}(x)$ and $u_1^{-1}(y)$, respectively, we get

$$f(\gamma x + (1-\gamma)y) = \theta f(x) + (1-\theta)f(y) \quad \text{for} \quad x, y \in u_1(I),\ x \geq y. \tag{43}$$

Let

$$D := \{(\gamma s, (1-\gamma)t) :\ s, t \in u_1(I),\ s \geq t\}.$$

Then, taking $(x, y) \in D$, we have $x = \gamma s$ and $y = (1-\gamma)t$ for some $s, t \in u_1(I)$ with $s \geq t$, and so applying (43) we obtain

$$f(x+y) = f(\gamma s + (1-\gamma)t) = \theta f(s) + (1-\theta)f(t) = \theta f\left(\frac{x}{\gamma}\right) + (1-\theta)f\left(\frac{y}{1-\gamma}\right).$$

Hence, taking $D_1 := \{\gamma s : s \in u_1(I)\}$ and $D_2 := \{(1-\gamma)t : t \in u_1(I)\}$, we get

$$f(x+y) = g(x) + h(y) \quad \text{for} \quad (x,y) \in D,$$

where $g : D_1 \to \mathbb{R}$ and $h : D_2 \to \mathbb{R}$ are given by $g(x) = \theta f\left(\frac{x}{\gamma}\right)$ for $x \in D_1$ and $h(y) = (1-\theta)f\left(\frac{y}{\gamma}\right)$ for $y \in D_2$, respectively. Note that the above equation is a Pexider equation on a restricted domain D. Moreover, as $u_1(I)$ is an interval, D is a connected subset of $\mathbb{R}^2$

with a nonempty interior. Furthermore

$$D_+ := \{x + y : (x, y) \in D\} = \{\gamma s + (1-\gamma)t,\ s, t \in u_1(I),\ s \geq t\} = u_1(I).$$

Therefore, applying the extension result of (Radó and Baker, 1987, Corollary 3), we obtain that there exists an additive mapping $a : \mathbb{R} \to \mathbb{R}$ and a $\beta \in \mathbb{R}$ such that $f(x) = a(x) + \beta$ for every x belonging to the interior of $u_1(I)$. Since f is continuous and strictly increasing, so is a, and hence, applying the standard argument (cf. e.g. Kuczma, 2008, Theorem 5.5.2), we get

$$f(x) = \alpha x + \beta \quad \text{for} \quad x \in u_1(I),$$

with some $\alpha \in (0, \infty)$. Thus, making use of (31), we obtain (42). Furthermore, inserting f into (43) gives

$$(\theta - \gamma)(x - y) = 0 \quad \text{for} \quad x, y \in u_1(I),\ x \geq y,$$

which yields $\theta = \gamma$ and completes the proof. □

Lemma 4 *Assume that for any z in the interior of X there exist a utility function $u_z : X_{\geq z} \to \mathbb{R}$ and a continuous probability weighting function w_z such that*

$$F(x, y; p) = u_z^{-1}\left(w_z(p)u_z(x) + (1 - w_z(p))u_z(y)\right) \quad \textit{for} \quad x \geq y \geq z,\ p \in [0, 1]. \tag{44}$$

Then $w_z = w_{z'} =: w$ for any z and z' in the interior of X and there exists a utility function $u : X \to \mathbb{R}$ such that (1) holds.

Proof. In view of (44), for any z and z' in the interior of X, with $z < z'$, any $x, y \in X$ such that $x \geq y \geq z'$ and every $p \in [0, 1]$, we have

$$u_z^{-1}\left(w_z(p)u_z(x) + (1 - w_z(p))u_z(y)\right) = u_{z'}^{-1}\left(w_{z'}(p)u_{z'}(x) + (1 - w_{z'}(p))u_{z'}(y)\right)$$

and so, according to Lemma 3, we get $w_z = w_{z'} =: w$ and

$$u_z(x) = \alpha u_{z'}(x) + \beta \quad \text{for} \quad x \geq z' \tag{45}$$

with some $\alpha \in (0, \infty)$ and $\beta \in \mathbb{R}$.

Let $(a_n : n \in \mathbb{N})$ be a decreasing sequence of elements of the interior of X such that $\lim_{n\to\infty} a_n = \inf X$. Moreover, let $a_0 \in X$ be such that $a_1 < a_0$. It follows from (44) that

$$u_{a_n}^{-1}(u_{a_n}(a_n) + w(p)(u_{a_n}(a_0) - u_{a_n}(a_n))) = F(a_0, a_n; p)$$
$$= u_{a_{n+1}}^{-1}(u_{a_{n+1}}(a_n) + w(p)(u_{a_{n+1}}(a_0) - u_{a_{n+1}}(a_n))) \quad \text{for} \quad n \in \mathbb{N},\ p \in [0, 1].$$

Furthermore, as w is a continuous probability weighting function, for every $n \in \mathbb{N}$ there is a unique $p_n \in (0,1)$ such that $w(p_n) = \frac{u_{a_n}(a_1)-u_{a_n}(a_n)}{u_{a_n}(a_0)-u_{a_n}(a_n)}$. Therefore, we have

$$\frac{u_{a_{n+1}}(a_1) - u_{a_{n+1}}(a_n)}{u_{a_{n+1}}(a_0) - u_{a_{n+1}}(a_n)} = \frac{u_{a_n}(a_1) - u_{a_n}(a_n)}{u_{a_n}(a_0) - u_{a_n}(a_n)} \quad \text{for} \quad n \in \mathbb{N}$$

and so

$$\frac{u_{a_{n+1}}(a_n) - u_{a_{n+1}}(a_1)}{u_{a_{n+1}}(a_0) - u_{a_{n+1}}(a_1)} = \frac{u_{a_n}(a_n) - u_{a_n}(a_1)}{u_{a_n}(a_0) - u_{a_n}(a_1)} \quad \text{for} \quad n \in \mathbb{N}.$$

Thus, since for any $n \in \mathbb{N}$, u_{a_n} is a utility function, in view of (47) we get

$$\lim_{x \to a_n^-} \frac{u_{a_{n+1}}(x) - u_{a_{n+1}}(a_1)}{u_{a_{n+1}}(a_0) - u_{a_{n+1}}(a_1)} = \frac{u_{a_{n+1}}(a_n) - u_{a_{n+1}}(a_1)}{u_{a_{n+1}}(a_0) - u_{a_{n+1}}(a_1)} = \frac{u_{a_n}(a_n) - u_{a_n}(a_1)}{u_{a_n}(a_0) - u_{a_n}(a_1)}. \tag{46}$$

Define a function $u : X \setminus \{\inf X\} \to \mathbb{R}$ in the following way

$$u(x) = u_{a_1}(x) \quad \text{for} \quad x \geq a_1,$$

$$u(x) = \frac{u_{a_{n+1}}(x) - u_{a_{n+1}}(a_1)}{u_{a_{n+1}}(a_0) - u_{a_{n+1}}(a_1)} \quad \text{for} \quad x \in [a_{n+1}, a_n),\ n \in \mathbb{N}. \tag{47}$$

Then, in view of (46)–(47), for any $n \in \mathbb{N}$, we have $\lim_{x \to a_n^-} u(x) = u(a_n)$, i.e. u is continuous on $[a_{n+1}, a_n]$ and so, it is continuous. Furthermore, as u is strictly increasing on $[a_{n+1}, a_n]$ for $n \in \mathbb{N}$, it is strictly increasing. Therefore, u is a utility function.

We show that (1) holds. To this end fix $x, y \in X \setminus \{\inf X\}$, with $x \geq y$, and $p \in [0,1]$. Let $m = \min\{n \in \mathbb{N} : a_n \leq y\}$, $k = \min\{n \in \mathbb{N} : a_n \leq F(x,y;p)\}$, and $l = \min\{n \in \mathbb{N} : a_n \leq x\}$. Since $y \leq F(x,y;p) \leq x$, we get $a_m \leq a_k \leq a_l$. Then making use of (45) we obtain that there exist $\alpha, \gamma \in (0,\infty)$ and $\beta, \delta \in \mathbb{R}$ such that

$$u_{a_l}(x) = \alpha u_{a_m}(x) + \beta \quad \text{for} \quad x \geq a_l, \tag{48}$$

$$u_{a_k}(x) = \gamma u_{a_m}(x) + \delta \quad \text{for} \quad x \geq a_k. \tag{49}$$

Hence, successively applying (47), (49), (44), (48) and again (47), we obtain

$$\begin{aligned}
u(F(x,y;p)) &= \frac{u_{a_k}(F(x,y;p)) - u_{a_k}(a_1)}{u_{a_k}(a_0) - u_{a_k}(a_1)} = \frac{u_{a_m}(F(x,y;p)) - u_{a_m}(a_1)}{u_{a_m}(a_0) - u_{a_m}(a_1)} \\
&= w(p)\frac{u_{a_m}(x) - u_{a_m}(a_1)}{u_{a_m}(a_0) - u_{a_m}(a_1)} + (1-w(p))\frac{u_{a_m}(y) - u_{a_m}(a_1)}{u_{a_m}(a_0) - u_{a_m}(a_1)} \\
&= w(p)\frac{u_{a_l}(x) - u_{a_l}(a_1)}{u_{a_l}(a_0) - u_{a_l}(a_1)} + (1-w(p))\frac{u_{a_m}(y) - u_{a_m}(a_1)}{u_{a_m}(a_0) - u_{a_m}(a_1)} \\
&= w(p)u(x) + (1-w(p))u(y)
\end{aligned}$$

□

The proofs of our main Theorems are divided into steps for better readability and clarity.

Proof of Theorem 1a. Note that uniqueness follows directly from Lemma 1. In the existence part, we only prove the sufficiency of the axioms, because their necessity is obvious. We assume that F satisfies **(Ref)**, **(CM)** and **(Perm)**.

Step 1. We show that since y_0 is the endpoint of X we may restrict attention to the case $y_0 = \min X$. In fact, suppose that in this case the representation (5) holds. Note that, if $y_0 = \max X$, then $\tilde{X} := \{2y_0 - x : \ x \in X\}$ is a real interval with $y_0 = \min \tilde{X}$ and a function $\tilde{F} : \Delta_{y_0}(\tilde{X}) \to \tilde{X}$, given by

$$\tilde{F}(x,p) = 2y_0 - F(2y_0 - x, p) \quad \text{for} \quad (x,p) \in \Delta_{y_0}(\tilde{X}),$$

satisfies **(Ref)**, **(CM)** and **(Perm)**. In fact, **(Ref)** and **(Perm)** are easy to verify and, because F is continuous and strictly increasing in payoff, $\tilde{F}$ has the same properties. Moreover, we have $2y_0 - x < y_0 < x$ for $x \in \tilde{X} \setminus \{y_0\}$, and so F is strictly decreasing in the probability of $2y_0 - x$. Hence, $\tilde{F}$ is strictly increasing in the probability of x and thus fulfills **(CM)**. We conclude that there exist a continuous probability weighting function w and a utility function $\tilde{u} : \tilde{X} \to \mathbb{R}$ satisfying $\tilde{u}(y_0) = 0$ such that

$$\tilde{F}(x,p) = \tilde{u}^{-1}(w(p)\tilde{u}(x)) \quad \text{for} \quad (x,p) \in \Delta_{y_0}(\tilde{X}).$$

Then $u : X \to \mathbb{R}$, defined by $u(x) = -\tilde{u}(2y_0 - x)$ for $x \in X$, is a utility function satisfying $u(y_0) = 0$ and for any $(x,p) \in \Delta_{y_0}(X)$, we get

$$F(x,p) = 2y_0 - \tilde{F}(2y_0 - x, p) = u^{-1}(-w(p)\tilde{u}(2y_0 - x)) = u^{-1}(w(p)u(x)),$$

that is the representation (5) holds. From now on we assume that $y_0 = \min X$.

Step 2. We show that for any $p \in (0,1)$, $F(x,p)$ is continuous and strictly increasing in x.

Let $p \in (0,1)$ and $x \in X$. First, assume that x is an interior point of X. Thus, taking $y \in X$ with $x < y$, we get $x = F(y,q)$ for some $q \in (0,1)$. Furthermore, for any sequence $(x_n : n \in \mathbb{N})$ sequence of elements of the interval (y_0, y) converging to x there exists a corresponding sequence $(q_n : n \in \mathbb{N})$ of elements of $(0,1)$ such that $x_n = F(y, q_n)$ for $n \in \mathbb{N}$. Thus, in view of **(CM)**, we get

$$F(y,q) = x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} F(y,q_n) = F(y, \lim_{n\to\infty} q_n)$$

and so $\lim_{n\to\infty} q_n = q$. Hence, making use of **(Perm)** and **(CM)**, we obtain

$$\lim_{n\to\infty} F(x_n, p) = \lim_{n\to\infty} F(F(y,q_n)),p) = \lim_{n\to\infty} F(F(y,p)),q_n)$$

$$= F(F(y,p),q) = F(F(y,q),p) = F(x,p).$$

If $x = y_0$ or $x = \max X$ then the same reasoning shows a right (left, respectively) continuity at x. This proves the continuity of $F(x,p)$ in x. We now prove the monotonicity. To this end fix $x_1, x_2 \in X$ with $y_0 \le x_1 < x_2$. Then, in view of **(CM)**, we get $x_1 = F(x_2, q)$ for some $q \in (0,1)$. Hence, applying **(Ref)**, **(CM)** and **(Perm)**, for every $p \in (0,1)$, we obtain

$$F(x_1,p) = F(F(x_2,q),p) = F(F(x_2,p),q) < F(F(x_2,p),1) = F(F(x_2,p)) = F(x_2,p).$$

Thus $F(x,p)$ is strictly monotone in x for any $p \in (0,1)$.

Step 3. We now derive the representation. If $\sup X \in X$ then put $b = \sup X$. Otherwise, let b be an arbitrary element of the interior of X. It follows from **(Ref)** that $F(b,0) = F(y_0) = y_0 < b = F(b) = F(b,1)$. Thus, in view of **(CM)**, for every $x \in [y_0, b]$ there exists a unique $p_x \in [0,1]$ such that $F(b,p_x) = x$. Applying the idea in Hosszú (1962) (cf. Aczél, 1966, pp. 270–271), define in $(y_0, b]$ a binary operation $\star$ in the following way

$$x \star y = F(y, p_x) \quad \text{for} \quad x, y \in (y_0, b]. \tag{50}$$

First, we show that $\star$ is commutative, associative, cancellative and continuous. To see that $\star$ is commutative, for any $x, y \in (y_0, b]$ apply **(Perm)**, to get

$$x \star y = F(y,p_x) = F(F(b,p_y),p_x) = F(F(b,p_x),p_y) = F(x,p_y) = y \star x.$$

Using commutativity of $\star$ and **(Perm)**, for every $x, y, z \in (y_0, b]$, we obtain

$$x \star (y \star z) = x \star (z \star y) = F(z \star y, p_x) = F(F(y,p_z),p_x)$$

$$= F(F(y, p_x), p_z) = F(x \star y, p_z) = z \star (x \star y) = (x \star y) \star z,$$

which proves that $\star$ is associative. To show the cancellativity of $\star$, suppose that $x \star z = y \star z$ for some $x, y, z \in (y_0, b]$. Then $F(z, p_x) = F(z, p_y)$ and so, taking **(Perm)** into account, we get

$$F(x, p_z) = F(F(b, p_x), p_z) = F(F(b, p_z), p_x) = F(z, p_x)$$

$$= F(z, p_y) = F(F(b, p_z), p_y) = F(F(b, p_y), p_z) = F(y, p_z).$$

Hence, in view of Step 2, we obtain $x = y$. In this way we have proved that $\star$ is right-cancellative. By commutativity, $\star$ is also left-cancellative and hence cancellative. Continuity of $\star$ follows from **(CM)** and Step 2. Thus we have proved that $\star$ possesses the required properties. Hence, applying Craigen and Páles (1989), we conclude that there exist an unbounded real interval I and a continuous bijection $f : (y_0, b] \to I$ such that

$$x \star y = f^{-1}(f(x) + f(y)) \quad \text{for} \quad x, y \in (y_0, b]. \tag{51}$$

Since replacing f by $-f$ does not alter (51), we may assume that f is strictly increasing. Note that $p_{F(b,p)} = p$ for $p \in [0, 1]$, and so it follows from (50) that

$$F(x, p) = F(b, p) \star x \quad \text{for} \quad x \in (y_0, b],\ p \in (0, 1].$$

Thus applying (51) on the right hand side yields

$$F(x, p) = f^{-1}(f(F(b, p)) + f(x)) \quad \text{for} \quad x \in (y_0, b],\ p \in (0, 1], \tag{52}$$

Setting $p = 1$ in (52), in view of **(Ref)**, we get

$$f(b) = 0. \tag{53}$$

Hence, as $f : (y_0, b] \to I$ is an increasing bijection and I is unbounded, we conclude that $I = (-\infty, 0]$ and $\lim_{x \to y_0^+} f(x) = -\infty$. Therefore, $u : [y_0, b] \to \mathbb{R}$ given by

$$u(x) = \begin{cases} e^{f(x)} & \text{for } x \in (y_0, b], \\ 0 & \text{for } x = y_0, \end{cases} \tag{54}$$

is a strictly increasing continuous function with $u(y_0) = 0$. Moreover, in view of **(CM)** and

(53), $w : [0,1] \to [0,1]$ defined by

$$w(p) = \begin{cases} e^{f(F(b,p))} & \text{for } p \in (0,1], \\ 0 & \text{for } p = 0, \end{cases} \tag{55}$$

is a continuous probability weighting function. From (52), (54) and (55) we derive that

$$F(x,p) = u^{-1}(w(p)u(x)) \quad \text{for} \quad x \in [y_0, b],\ p \in [0,1].$$

If $\sup X \in X$, this gives a required representation. If $\sup X \notin X$, then as b is an arbitrary element of $[y_0, a)$, applying Lemma 2, we get the assertion. □

In the sequel, we will use the following notation: $X_{\leq y_0} := X \cap (-\infty, y_0]$ and $X_{\geq y_0} := X \cap [y_0, \infty)$. Similarly, we set $X_{<y_0} := X \cap (-\infty, y_0)$ and $X_{>y_0} := X \cap (y_0, \infty)$.

Proof of Theorem 1b. If $F : \Delta_{y_0}(X) \to X$ satisfies **(Ref)**, **(CM)** and **(Perm)**, then applying Theorem 1a twice (first with X replaced by $X_{\leq y_0}$, and then with X replaced by $X_{\geq y_0}$), we obtain the existence of continuous probability weighting functions w_-, w_+ and utility functions $u_- : X_{\leq y_0} \to \mathbb{R}$ and $u_+ : X_{\geq y_0} \to \mathbb{R}$ such that $u_-(y_0) = u_+(y_0) = 0$ and

$$F(x,p) = \begin{cases} u_-^{-1}(w_-(p)u_-(x)) & \text{for } x < y_0,\ p \in [0,1], \\ u_+^{-1}(w_+(p)u_+(x)) & \text{for } x \geq y_0,\ p \in [0,1]. \end{cases}$$

This yields the representation (2) with $u : X \to \mathbb{R}$ given by

$$u(x) = \begin{cases} u_-(x) & \text{for } x < y_0, \\ u_+(x) & \text{for } x \geq y_0. \end{cases}$$

Note also that, as u_- and u_+ are utility functions with $u_+(y_0) = u_-(y_0) = 0$, u is a utility function with $u(y_0) = 0$. This completes the existence part of the proof.

We now prove uniqueness. Assume that (2) is satisfied with w_-, w_+ replaced by another pair of probability weighting functions $\tilde{w}_-, \tilde{w}_+$, and u replaced by another utility function $\tilde{u} : X \to \mathbb{R}$ satisfying $\tilde{u}(y_0) = 0$. Let u_- and u_+ be the restrictions of u to $X_{\leq y_0}$ and $X_{\geq y_0}$, respectively. Similarly, let $\tilde{u}_-$ and $\tilde{u}_+$ be the corresponding restrictions of $\tilde{u}$. Then we get

$$\begin{aligned} \tilde{u}_-^{-1}(\tilde{w}_-(p)\tilde{u}_-(x)) &= u_-^{-1}(w_-(p)u_-(x)) \quad \text{for} \quad x \in X_{\leq y_0},\ p \in [0,1], \\ \tilde{u}_+^{-1}(\tilde{w}_+(p)\tilde{u}_+(x)) &= u_+^{-1}(w_+(p)u_+(x)) \quad \text{for} \quad x \in X_{\leq y_0},\ p \in [0,1]. \end{aligned}$$

Therefore, applying Lemma 1, we obtain that there exist $\alpha, \beta, r_-, r_+ \in (0, \infty)$ such that (8) and (9) hold, $\tilde{u}_-(x) = -\alpha(-u_-(x))^{r_-}$ for $x \in X_{\leq y_0}$ and $\tilde{u}_+(x) = \beta u_+(x)^{r_+}$ for $x \in X_{\geq y_0}$. Hence, (10) holds which concludes the uniqueness part of the proof of Theorem 1b. □

Proof of Theorem 2. The uniqueness part follows from the uniqueness parts of Theorems 1a and 1b in the respective two cases, with the additional restriction that the weighting functions acting in these theorems are the identity on $[0,1]$. We now prove the existence part. Necessity of the axioms is obvious. In order to prove their sufficiency assume that **(Ref)**, **(CM)** and **(Red)** hold. First assume that y_0 is the endpoint of X. Then, as **(Red)** implies **(Perm)**, applying Theorem 1a there exist a continuous probability weighting function w and a utility function $u : X \to \mathbb{R}$ satisfying $u(y_0) = 0$ such that F is of the form (5). Plugging it into **(Red)** we obtain

$$w(pq) = w(p)w(q) \quad \text{for} \quad p, q \in [0,1].$$

Hence, since w is continuous, by the standard result (see for example Kuczma, 2008, Theorem 13.1.6) there exists $\alpha > 0$ such that $w(p) = p^\alpha$ for $p \in [0,1]$. Therefore, defining $\tilde{u} : X \to \mathbb{R}$ by $\tilde{u}(x) = u(x)^{1/\alpha}$ for $x \in X$, and taking (5) into account, we conclude that $F(x,p) = \tilde{u}^{-1}(p\tilde{u}(x))$ for $x \in X$ and $p \in [0,1]$. This yields the required representation in the case when y_0 is the endpoint of X. If y_0 is the interior point of X then, as **(Red)** implies **(Perm)**, according to Theorem 1b, F is of the form (2) with some continuous probability weighting functions w_-, w_+, and a utility function $u : X \to \mathbb{R}$ satisfying $u(y_0) = 0$. Similarly as before we thus obtain that

$$w_i(pq) = w_i(p)w_i(q) \quad \text{for} \quad p, q \in [0,1],\ i \in \{+,-\},$$

which yields that $w_i(p) = p^{\alpha_i}$, $p \in [0,1]$ for some $\alpha_i > 0$. Hence the required representation holds with $\tilde{u} : X \to \mathbb{R}$ given by

$$\tilde{u}(x) = \begin{cases} -(-u(x))^{1/\alpha_-} & \text{for} \quad x \leq y_0, \\ u(x)^{1/\alpha_+} & \text{for} \quad x \geq y_0. \end{cases} \tag{56}$$

□

Proof of Theorem 3. The 'if' part of the uniqueness is straightforward. The 'only if' part follows directly from Lemma 3. That the axioms are necessary for the representation is clear.

We now prove their sufficiency. Assume that **(Ref)**, **(CM)** and **(Dist)** hold. Let y_1 be an arbitrarily fixed element of the interior of X. The remaining part of the proof is divided into three steps. In Step 1, we show that the axioms imply a special case of the functional equation analyzed by Gilányi et al. (2005). In Step 2, we use their solution to establish a required representation for prospects with payoffs in $X_{\geq y_1}$. Finally, the representation for arbitrary prospects in $\Delta(X)$ is derived in Step 3.

Step 1. Let y_0 be an arbitrary element of X such that $y_0 < y_1$. For $i \in \{0,1\}$ define the function $F_i : \Delta_{y_i}(X_{\geq y_i}) \to X_{\geq y_i}$ by

$$F_i(x;p) = F(x, y_i; p) \quad \text{for} \quad (x;p) \in \Delta_{y_i}(X_{\geq y_i}). \tag{57}$$

Setting in **(Dist)** $y = z = y_i$ for $i \in \{0,1\}$, in view of **(Ref)**, we get

$$F_i(F_i(x,p);q) = F_i(F_i(x,q);p) \quad \text{for} \quad x \in X_{\geq y_i},\ p \in [0,1],\ i \in \{0,1\}. \tag{58}$$

Thus, making use of **(CM)** and applying Theorem 1a, we obtain that for $i \in \{0,1\}$ there exist a continuous probability weighting function w_i and a utility function $v_i : X_{\geq y_i} \to \mathbb{R}$ such that $v_i(y_i) = 0$ and

$$F_i(x;p) = v_i^{-1}(w_i(p)v_i(x)) \quad \text{for} \quad x \in X_{\geq y_i},\ p \in [0,1],\ i \in \{0,1\}. \tag{59}$$

Hence $v_0(y_1) > v_0(y_0) = 0$ and so, normalizing v_0, we conclude that for $\bar{v}_0 := \frac{v_0}{v_0(y_1)}$, we have

$$\bar{v}_0(y_0) = 0, \quad \bar{v}_0(y_1) = 1 \tag{60}$$

and

$$F_0(x;p) = \bar{v}_0^{-1}(w_0(p)\bar{v}_0(x)) \quad \text{for} \quad x \in X_{\geq y_0},\ p \in [0,1]. \tag{61}$$

Furthermore, setting in **(Dist)** $z = y_0$, in view of (57) and (61), we get

$$\bar{v}_0^{-1}(w_0(q)\bar{v}_0(F(x,y;p))) = F(\bar{v}_0^{-1}(w_0(q)\bar{v}_0(x)), \bar{v}_0^{-1}(w_0(q)\bar{v}_0(y)); p)$$

for $x, y \in X_{\geq y_0}$ with $x \geq y$ and $p, q \in [0,1]$. Replacing in this equality x and y by $\bar{v}_0^{-1}(x)$ and $\bar{v}_0^{-1}(y)$, respectively, we conclude that

$$W_p(w_0(q)x, w_0(q)y) = w_0(q)W_p(x,y) \ \text{ for } \ x, y \in \bar{v}_0(X_{\geq y_0}),\ x \geq y,\ p, q \in [0,1], \tag{62}$$

where for any $p \in [0,1]$ a function $W_p : \bar{v}_0(X_{\geq y_0})^2 \to \mathbb{R}$ is given by

$$W_p(x,y) = \bar{v}_0(F(\bar{v}_0^{-1}(x), \bar{v}_0^{-1}(y); p)) \quad \text{for} \quad x, y \in \bar{v}_0(X_{\geq y_0}). \tag{63}$$

Since w_0 is a continuous probability weighting function, we have $\{w_0(p) : p \in [0,1]\} = [0,1]$

and so it follows from (62) that

$$W_p(\lambda x, \lambda y) = \lambda W_p(x,y) \quad \text{for} \quad x, y \in \bar{v}_0(X_{\geq y_0}),\ x \geq y,\ \lambda \in [0,1],\ p \in [0,1]. \tag{64}$$

Moreover, in view of (60), we have $\frac{1}{\bar{v}_0(y)} \in (0,1]$ for $y \in X_{\geq y_1}$. Hence, applying (64), for every $x, y \in X_{\geq y_1}$ with $x \geq y$ and $p \in [0,1]$, we obtain

$$W_p(\bar{v}_0(x), \bar{v}_0(y)) = \bar{v}_0(y) \frac{1}{\bar{v}_0(y)} W_p(\bar{v}_0(x), \bar{v}_0(y)) = \bar{v}_0(y) W_p\left(\frac{\bar{v}_0(x)}{\bar{v}_0(y)}, 1\right).$$

Thus, taking (63) into account, for every $x, y \in X_{\geq y_1}$, with $x \geq y$, and $p \in [0,1]$, we get

$$F(x,y;p) = \bar{v}_0^{-1}(W_p(\bar{v}_0(x), \bar{v}_0(y))) = \bar{v}_0^{-1}\left(\bar{v}_0(y) W_p\left(\frac{\bar{v}_0(x)}{\bar{v}_0(y)}, 1\right)\right).$$

Hence

$$F(x,y;p) = \bar{v}_0^{-1}\left(\bar{v}_0(y) \Phi_p\left(\frac{\bar{v}_0(x)}{\bar{v}_0(y)}\right)\right) \quad \text{for} \quad x, y \in X_{\geq y_1},\ x \geq y,\ p \in [0,1], \tag{65}$$

where, for every $p \in [0,1]$, a function $\Phi_p : I \to \mathbb{R}$ is given by

$$\Phi_p(s) = W_p(s, 1) \quad \text{for} \quad s \in I, \tag{66}$$

with $I := \bar{v}_0(X_{\geq y_1})$. Note that, as $\bar{v}_0$ is a utility function and $\bar{v}_0(y_1) = 1$, I is a real interval containing its left endpoint 1. Furthermore, plugging (65) into **(Dist)** with $z = y_1$, in view of (57) and (60), for every $x, y \in X$, with $x \geq y \geq y_1$ and $p, q \in [0,1]$, we get

$$\Phi_q\left(\bar{v}_0(y) \Phi_p\left(\frac{\bar{v}_0(x)}{\bar{v}_0(y)}\right)\right) = \Phi_q\left(\bar{v}_0(y)\right) \Phi_p\left(\frac{\Phi_q\left(\bar{v}_0(x)\right)}{\Phi_q\left(\bar{v}_0(y)\right)}\right).$$

Hence, we have

$$\frac{\Phi_q\left(t \Phi_p\left(\frac{s}{t}\right)\right)}{\Phi_q(t)} = \Phi_p\left(\frac{\Phi_q(s)}{\Phi_q(t)}\right) \quad \text{for} \quad s, t \in I,\ s \geq t,\ p, q \in [0,1]. \tag{67}$$

Since $\bar{v}_0(y_1) = 1$, applying (66), (63), (57) and (58) successively, we obtain

$$\begin{aligned} \Phi_p(s) &= W_p(s,1) = \bar{v}_0(F(\bar{v}_0^{-1}(x), \bar{v}_0^{-1}(1); p)) = \bar{v}_0(F(\bar{v}_0^{-1}(s), y_1; p)) \\ &= \bar{v}_0(F_1(\bar{v}_0^{-1}(s); p)) = \bar{v}_0(v_1^{-1}(w_1(p) v_1(\bar{v}_0^{-1}(s)))) \quad \text{for} \quad s \in I,\ p \in [0,1]. \end{aligned}$$

Therefore, setting $\Phi := v_1 \circ \bar{v}_0^{-1}$, we get

$$\Phi_p(s) = \Phi^{-1}(w_1(p)\Phi(s)) \quad \text{for} \quad s \in I,\ p \in [0,1]. \tag{68}$$

Note that, as I is an interval containing its left endpoint 1, the interior of I is of the form $(1, d)$ with some $1 < d \leq \infty$. We show that for any $q \in [0,1]$ and $t \in (1,d)$ a function $f_{(q,t)} : \left(1, \frac{d}{t}\right) \to \mathbb{R}$ defined in the following way

$$f_{(q,t)}(x) = \frac{1}{\Phi(x)} \Phi\left(\frac{\Phi_q(tx)}{\Phi_q(t)}\right) \quad \text{for} \quad x \in \left(1, \frac{d}{t}\right), \tag{69}$$

is constant. Fix $q \in [0,1]$, $t \in (1,d)$ and $x_1, x_2 \in \left(1, \frac{d}{t}\right)$ with $x_1 < x_2$. Let $s \in (tx_2, d)$. Then $x_i < \frac{s}{t}$ for $i \in \{1,2\}$ and so, as Φ is strictly increasing, with $\Phi(1) = (v_1 \circ \bar{v}_0^{-1})(1) = v_1(y_1) = 0$, we have $\frac{\Phi(x_i)}{\Phi\left(\frac{s}{t}\right)} \in (0,1)$ for $i \in \{1,2\}$. Thus, since w_1, being a continuous probability weighting function, is onto $[0,1]$, for $i \in \{1,2\}$ there exists $p_i \in (0,1)$ such that $w_1(p_i) = \frac{\Phi(x_i)}{\Phi\left(\frac{s}{t}\right)}$. Hence, in view of (68), for $i \in \{1,2\}$, we get

$$\Phi_{p_s}\left(\frac{s}{t}\right) = \Phi^{-1}\left(w_1(p_1)\Phi\left(\frac{s}{t}\right)\right) = x_i$$

and

$$(\Phi \circ \Phi_{p_i})\left(\frac{\Phi_q(s)}{\Phi_q(t)}\right) = w_1(p_i)\Phi\left(\frac{\Phi_q(s)}{\Phi_q(t)}\right) = \frac{\Phi\left(\frac{\Phi_q(s)}{\Phi_q(t)}\right)}{\Phi\left(\frac{s}{t}\right)}\Phi(x_i).$$

Applying Φ on both sides of (67) with $p = p_i$, and making use of the above two equalities we obtain

$$\frac{1}{\Phi(x_i)}\Phi\left(\frac{\Phi_q(tx_i)}{\Phi_q(t)}\right) = \frac{\Phi\left(\frac{\Phi_q(s)}{\Phi_q(t)}\right)}{\Phi\left(\frac{s}{t}\right)} \quad \text{for} \quad i \in \{1,2\}.$$

Thus, taking (69) into account, we conclude that $f_{(q,t)}(x_1) = f_{(q,t)}(x_2)$, and hence $f_{(q,t)}$ is constant, say $f_{(q,t)}(x) = c(q,t)$ for $x \in \left(1, \frac{d}{t}\right)$, with some $c(q,t) \in \mathbb{R}$. So, in view of (69), for any $q \in [0,1]$, $t \in (1,d)$ and $x \in \left(1, \frac{d}{t}\right)$, we have

$$c(q,t) = \frac{1}{\Phi(x)}\Phi\left(\frac{\Phi_q(tx)}{\Phi_q(t)}\right) > \frac{\Phi(1)}{\Phi(x)} = 0$$

and

$$\Phi\left(\frac{\Phi_q(tx)}{\Phi_q(t)}\right) = c(q,t)\Phi(x).$$

Therefore, for any $q \in [0,1]$, we obtain

$$\Psi(H_q(\ln t) - H_q(\ln t + \ln x)) = G_q(\ln t) + \Psi(\ln x) \quad \text{for} \quad t, x \in (1,d),\ tx \in (1,d),$$

where $G_q, H_q, \Psi : (0, \ln d) \to \mathbb{R}$ are given by

$$H_q(z) = -\ln \Phi_q(e^z) \quad \text{for} \quad z \in (0, \ln d), \tag{70}$$

$$G_q(z) = \ln c(q, e^z) \quad \text{for} \quad z \in (0, \ln d), \tag{71}$$

$$\Psi(y) = \ln \Phi(e^y) \quad \text{for} \quad y \in (0, \ln d), \tag{72}$$

with a convention $\ln \infty = \infty$. Thus, for every $q \in [0,1]$, we have

$$H_q(z) - H_q(z+y) = \Psi^{-1}\left(G_q(z) + \Psi(y)\right) \quad \text{for} \quad z, y \in (0, \ln d),\ z+y \in (0, \ln d). \tag{73}$$

Note that, since Φ and Φ_q for $q \in (0,1)$ are continuous, it follows from (70) and (72) that so are Ψ and H_q for $q \in (0,1)$. Thus, in view of (73), G_q is continuous for any $q \in (0,1)$.

Step 2. Equation (73) is a particular case of the functional equation analyzed by Gilányi et al. (2005). Thus, according to their Theorem 2, for every $q \in [0,1]$, G_q is either constant or it is strictly monotone. We will first consider the case where G_q is strictly monotone for some $q \in [0,1]$, and then the case where G_q is constant for every $q \in [0,1]$.

Assume that G_q is strictly monotone for some $q \in [0,1]$. Then, as Ψ is strictly increasing, according to (Gilányi et al., 2005, Theorem 2), either there exist $\alpha \in \mathbb{R} \setminus \{0\}$, $\beta \in (0, \infty)$ and $\gamma \in \mathbb{R}$ such that

$$\Psi(x) = \beta \ln \left|1 - e^{-\alpha x}\right| + \gamma \quad \text{for} \quad x \in (0, \ln d), \tag{74}$$

or there exist $\beta \in (0, \infty)$ and $\gamma \in \mathbb{R}$ such that

$$\Psi(x) = \beta \ln x + \gamma \quad \text{for} \quad x \in (0, \ln d). \tag{75}$$

If (74) holds, then as ϕ is continuous with $\Phi(1) = 0$, in view of (72), we get

$$\Phi(x) = e^{\gamma} \left|1 - x^{-\alpha}\right|^{\beta} \quad \text{for} \quad x \in I.$$

Therefore, considering separately the case of α negative and then α positive and in each of them applying first (68) and then (65), for every $x, y \in X_{\geq y_1}$ with $x \geq y$ and $p \in [0,1]$, we get

$$F(x,y;p) = \bar{v}_0^{-1}\left(\left(w_1(p)^{\frac{1}{\beta}}\bar{v}_0(x)^{-\alpha} + (1-w_1(p)^{\frac{1}{\beta}})\bar{v}_0(y)^{-\alpha}\right)^{-\frac{1}{\alpha}}\right). \tag{76}$$

Define $w := w_1^{\frac{1}{\beta}}$ and $u_0 : X_{\geq y_1} \to \mathbb{R}$ as

$$u_0(x) = |\bar{v}_0(x)^{-\alpha} - 1| \quad \text{for} \quad x \in X_{\geq y_1}.$$

Then w is a continuous probability weighting function and u_0 is a utility function with $u_0(y_1) = 0$. Furthermore, in view of (76), we have

$$F(x,y;p) = u_0^{-1}\left(w(p)u_0(x) + (1-w(p))u_0(y)\right) \quad \text{for} \quad x \geq y \geq y_1,\ p \in [0,1]. \tag{77}$$

If (75) holds, then the functional equation (73) becomes

$$H_q(z) - H_q(z+y) = e^{\frac{1}{\beta}G_q(z)}y \quad \text{for} \quad z, y \in (0, \ln d),\ z + y < \ln d. \tag{78}$$

Since H_q is continuous and $H_q(0) = 0$, it follows from (78) that

$$\frac{H_q(y)}{y} = -\lim_{z\to 0^+} e^{\frac{1}{\beta}G_q(z)} =: a \quad \text{for } y \in (0, \ln d).$$

Therefore $H_q(y) = ay$ for $y \in (0, \ln d)$ and so from (78) we derive that $e^{\frac{1}{\beta}G_q(z)} = -a$ for $z \in (0, \ln d)$, which contradicts the strict monotonicity of G_q.

We now consider the case where G_q is constant for every $q \in [0,1]$. Then, as for every $q \in [0,1]$, H_q is a strictly decreasing continuous function with $H_q(0) = 0$, applying again (Gilányi et al., 2005, Theorem 2), we conclude that for every $q \in [0,1]$ there exists a $w(q) \in (0,\infty)$ such that $H_q(x) = -w(q)x$ for $x \in (0, \ln d)$. Thus, in view of (70) and the fact that Φ_q is continuous for every $q \in [0,1]$, we get

$$\Phi_q(x) = x^{w(q)} \quad \text{for} \quad x \in I,\ q \in [0,1]. \tag{79}$$

Hence, taking (65) into account, we obtain

$$F(x,y;p) = \bar{v}_0^{-1}\left(\bar{v}_0(x)^{w(p)}\bar{v}_0(y)^{1-w(p)}\right) \quad \text{for} \quad x \geq y \geq y_1,\ p \in [0,1].$$

This yields (77) with $u_0 : X_{\geq y_1} \to \mathbb{R}$ given by $u_0(x) = \ln \bar{v}_0(x)$ for $x \in X_{\geq y_1}$. Note that u_0 is a utility function, with $u_0(y_1) = 0$. Moreover, it follows from **(CM)** and (77) that w is a continuous probability weighting function.

This concludes the analysis of all possible cases. In one of them a contradiction was derived, while in the other two we obtained the required representation (77).

Step 3. Since y_1 was an arbitrary element in the interior of X, applying Lemma 4, we conclude that if $\inf X \notin X$ the representation (1) holds with some utility function u and a continuous probability weighting function w.

Assume that $\ell := \inf X \in X$. Then, in view of **(CM)**, for every $x \in X \setminus \{\ell\}$ and $p \in (0,1)$, we have $F(x,\ell;p) \in X \setminus \{\ell\}$ and so we get

$$c := \lim_{y \to \ell^+} u(y) = \lim_{y \to \ell^+} \frac{u(F(x,y;p)) - w(p)u(x)}{1 - w(p)} = \frac{u(F(x,\ell;p)) - w(p)u(x)}{1 - w(p)}.$$

Therefore, extending u to X by putting $u(\ell) = c$, we conclude that u is a utility function on X. Furthermore, for every $x \in X \setminus \{\ell\}$ and $p \in (0,1)$, we have

$$F(x,\ell;p) = u^{-1}(w(p)u(x) + (1 - w(p))c) = u^{-1}(w(p)u(x) + (1 - w(p))u(\ell)).$$

Obviously, in view of **(Ref)**, the last equality holds also for $x = \ell$ or $p \in \{0,1\}$. Thus, a proof of the representation (1) is completed. □

Proof of Corollary 1. Necessity of the axioms is obvious. We now prove their sufficiency. Assume that **(Ref)**, **(CM)**, and **($\overline{\textbf{Dist}}$)** hold. Since **($\overline{\textbf{Dist}}$)** implies **(Dist)**, by Theorem 3 we obtain the existence of a utility function u and a continuous probability weighting function w such that (1) holds. In view of (14), in order to get the required representation, it is enough to show that w satisfies (15), i.e. it is self-conjugate. Fix $x, y, z \in X$ with $x > z > y$ and $q \in [0,1]$. According to **(CM)** and **(Ref)** there exists $p \in (0,1)$ such that $F(x,y;p) > z$. Then applying (1), we get

$$F(F(x,y;p),z;q) = u^{-1}(w(q)w(p)u(x) + w(q)(1 - w(p))u(y) + (1 - w(q))u(z)).$$

Furthermore, using **(CM)** and **(Ref)** again, we obtain

$$y \le F(y,z;q) \le z \le F(x,z;q) \le x.$$

Thus, applying (1) and (14), yields

$$F(F(x,z;q),F(y,z;q);p)$$

$$= u^{-1}(w(p)w(q)u(x) + (1 - w(1-q))(1 - w(p))u(y) + (w(p)(1 - w(q)) + (1 - w(p)w(1-q))u(z)).$$

Therefore, in view of **($\overline{\textbf{Dist}}$)** we get

$$(1 - w(p))(w(q) + w(1 - q) - 1)(u(z) - u(y)) = 0.$$

Since the first and the third part of the above product are strictly positive, the middle part must be zero. Since q is arbitrary, this proves that w is self-conjugate, which ends the proof. □

Proof of Theorem 4. The uniqueness part is standard. In fact, it follows from Theorem 3. That the axioms are necessary for the existence of the representation is obvious. We now prove their sufficiency.

Assume that $F : \Delta(X) \to X$ satisfies **(Ref)**, **(CM)** and **(Red2)**. Note that **(Red2)** is a system of **(Red)** indexed by $y \in X$. Therefore, according to Theorem 2, for any $y \in X$, there exists a utility function $u_y : X \to \mathbb{R}$ such that $u_y(y) = 0$ and

$$F(x, y; p) = u_y^{-1}(pu_y(x)) \quad \text{for} \quad x \in X,\ p \in [0, 1]. \tag{80}$$

If $\inf X \in X$ then put $z = \inf X$. Otherwise, let z be an arbitrary element of the interior of X. Then, setting $u := u_z$, in view of (4) and (80), we get

$$u_y^{-1}(pu_y(z)) = F(z, y; p) = F(y, z; 1 - p) = u^{-1}((1 - p)u(y)) \quad \text{for} \quad y \in X,\ p \in [0, 1]. \tag{81}$$

Since $1 - \frac{u(x)}{u(y)} \in (0, 1]$ for $x, y \in X$ with $y > x \geq z$, applying (80) in the first equality and (81) in the third and fifth equalities, we obtain

$$\begin{aligned} F(x, y; p) &= u_y^{-1}(pu_y(x)) \\ &= u_y^{-1}\left(pu_y\left(u^{-1}\left(\left(1 - \left(1 - \tfrac{u(x)}{u(y)}\right)\right)u(y)\right)\right)\right) \\ &= u_y^{-1}\left(pu_y\left(u_y^{-1}\left(\left(1 - \tfrac{u(x)}{u(y)}\right)u_y(z)\right)\right)\right) \\ &= u_y^{-1}\left(p\left(1 - \tfrac{u(x)}{u(y)}\right)u_y(z)\right) \\ &= u^{-1}\left(\left(1 - p\left(1 - \tfrac{u(x)}{u(y)}\right)\right)u(y)\right) \\ &= u^{-1}(pu(x) + (1 - p)u(y)). \end{aligned}$$

Thus, in view of **(Ref)**, we get

$$F(x, y; p) = u^{-1}(pu(x) + (1 - p)u(y)) \quad \text{for} \quad x, y \in X_{\geq z},$$

which concludes the proof in the case $\inf X \in X$. If $\inf X \notin X$, then the assertion follows from Lemma 4. □

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