Biseparable representations of the Certainty Equivalents

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Abstract

We consider the following biseparable representation of the certainty equivalent: $F(x, y; p) = u^{-1}(w(p)u(x) + (1 - w(p))u(y))$, where $(x, y; p)$ is the binary monetary prospect, u is the utility function, and w is the probability weighting function. We provide a simple set of axioms characterizing this form for all binary prospects as well as for the subset of binary prospects, called simple prospects, in which one of the two payoffs is fixed. We consider both the case of general w and the case of expected utility, where w is the identity function. We discuss the extent to which such models can be identified, the issue of extending these models to a larger number of payoffs, and draw conclusions for model testing.

Keywords: biseparable model, rank-dependence, expected utility, certainty equivalents

JEL codes: D81, D90

1 Introduction

What we do Let $(x, y; p)$ denote a risky prospect that pays x dollars with probability p, and y with probability $1 - p$, and let $F(x, y; p)$ be its certainty equivalent (CE), i.e. the sum of money for which, in a choice between the money and the prospect, the decision maker is indifferent between the two. In this article, we are interested in individual preferences that lead to the following biseparable model of the certainty equivalent

$$F(x, y; p) = u^{-1}(w(p)u(x) + (1 - w(p))u(y)) \quad \text{for} \quad x \geq y, \ p \in [0, 1]. \tag{1}$$

where u is a utility function and w a probability weighting function.

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We consider several models in this class. In particular, we characterize (1) in the domain of all binary prospects, as well as in the subset of simple prospects, i.e., prospects in which one of the two payoffs is fixed. For both domains, we consider the case of a general probability weighting function w (corresponding to the CE in the rank-dependent utility model) and the special case where w is the identity function (CE in the expected utility model). We also analyze intermediate cases in which w satisfies the condition $w(0.5) = 0.5$ (CE in the anticipated utility model of Quiggin, 1982) or a stronger self-conjugacy condition (CE in the rank-independent utility model). Our characterizations are based on novel axioms. We discuss their relationship to existing axioms and characterizations. In particular, we comprehensively state the relation of our key distributivity axiom to known and new axioms based on the bisymmetry axiom of Aczél (1947). Below we justify why it is worth considering the model (1) and its special cases.

Preference class The preferences generating CE of the form (1) are very general. In particular, as shown in section 4.4, they are consistent with the phenomenon of preference reversals (Grether and Plott, 1979). Even if we rule out such inversions by assuming transitivity and monotonicity, these preferences, which can then be represented by certainty equivalents, are still very general. They consist of all preferences that, on the set of binary prospects, match the binary rank-dependent utility model. [1] Such preferences include many popular nonexpected preference models, such as, for example, (Wakker, 2010, Observation 7.11. 1): the rank-dependent utility model (Quiggin, 1982; Chew and Epstein, 1989), the RAM and TAX model (Birnbaum, 2008), the disappointment aversion theory (Gul, 1991), the original prospect theory restricted to gains or losses (Kahneman and Tversky, 1979), or the prospective reference theory (Viscusi, 1989). By focusing on the (1) model, we examine the "common denominator" for all these models.

Model identification vs. model validation. A common practice in descriptive and prescriptive approaches to decision support is to "train" a model on a training data set and then use it to make predictions on a test data set. In the case of a preference model, the data is provided by human participants in preference elicitation experiments. This data must be sufficient to identify the model parameters. For example, a rank-dependent utility model for finite-support prospects can be fully identified using certainty equivalents of binary prospects. However, data for only simple prospects (binary prospects with a fixed payoff) are insufficient. In section 4.1 we discuss the implications of some of our results for model identification on the set of simple and binary prospects.

Most popular models are identifiable on a small subset of their domain. Thus, in order to reduce the cognitive load on experimental subjects and minimize noise, experimental designers most often choose low-complexity task sets. Often, model parameters are identified using certainty equivalents rather than choice data, because the former provide point rather than interval estimates. For example, Tversky and Kahneman (1992) uses exclusively the certainty equivalents (CEs) of binary prospects, mostly with one common payoff (simple

[1] Luce and Narens (1985) investigate the concepts of m-point homogeneity and n-point uniqueness for general scales and find that rank-dependent utility is the most general interval scale for two states of nature. See also Sokolov (2011).

prospects), to identify the parameters of cumulative prospect theory (see also Gonzalez and Wu, 1999). Similarly, in the normative application of the theory, the decision maker is asked to make a series of simple choices (e.g., determine the CEs of binary prospects) in order to identify a given utility model, which is ultimately used to predict more complex choices, e.g., choosing between prospects with multiple payoffs (Gilboa, 2009, p .87, see also Luce and Raiffa, 1957, section 2.8).

This approach, in which data from low-complexity tasks are used as input to a model to infer more complex choices, is reasonable if we know (assume) that the model is true in its domain. This is the case for model identification. However, for model validation we need data drawn from the entire domain. For example, we cannot use data exclusively for binary prospects to validate a rank-dependent utility model for prospects with multiple payoffs. These data verify the quality of the model only on the set of binary prospects. Some authors seem to forget this, judging the quality of a model on the entire domain based on the quality of the model fit on a subdomain. Since the vast majority of experimental data on choice under risk involve data on choice between binary prospects or their CEs, we can only make confident judgments about the quality of the model for binary prospects. There is no similar consensus for models whose domain includes prospects with more than two payoffs, because there is much less data on such prospects and the existing data are not sufficiently conclusive. This further justifies our interest in the biseparable model.

Cross-domain model extension Köbberling and Wakker (2003) discusses ways of deriving characteristizations of a binary rank-dependent utility model as special cases of a general rank-dependent utility model using trade-off techniques. This approach of deriving a binary model from a general model, of which the binary model is a special case, has some drawbacks. In particular, axioms that can be stated for any finite number of payoffs often cannot be directly stated for a fixed number of them. For example, the well-known characterizations of the quasi-arithmetic mean, the counterpart of CE, (Nagumo, 1930; Kolmogorov, 1930; de Finetti, 1931) rely on axioms requiring that the means be defined for an arbitrary number of payoffs. The replacement axiom of Kolmogorov (1930) written for two payoffs reduces to a tautology. In turn, the quasilinearity axiom of de Finetti (1931) uses the idea of a mixture of two probability distributions. A mixture of two probability distributions with finite support also has finite support. However, mixing two binary distributions does not have to be binary. In section 4.3 we show that writing the axiom of quasilinearity in the domain of binary prospects (so that the mixtures are also binary) can be nontrivial. Moreover, we show that even such a binary version of this axiom is still much stronger than our reduction axiom, which was used to characterize the corresponding model of the certainty equivalent of the binary expected utility model.

Our approach is based on the characterization concept of Aczél (1947), which instead of the mean for any number of payoffs, characterized the mean for one specific number of payoffs. Similarly, we identify minimal axioms for a given domain. In this way, we provide tools for precisely addressing the problem of extending a given model from a restricted domain to a broader one. For example, we can specify precisely and by how much the axioms need to be strengthened to extend a biseparable model from the domain of simple prospects to binary prospects, or a certainty equivalent model for rank-dependent utility from the domain of binary prospects to prospects with multiple payoffs. We address such questions in the discussion that follows our main results in sections 2.1 and 2.2. In section 4.2 we also discuss the same issue from the point of view of representations rather than axioms, that is, we discuss nonbiseparable extensions of a biseparable model.

There are some works that characterize (1) or binary rank-dependent utility model (Pfanzagl, 1959; Miyamoto, 1988; Luce, 1991; Luce and Fishburn, 1991). Compared to them, our work offers characterizations based on new key axioms for the most important cases of biseparable utility model (rank-dependent, rank-independent, anticipated utility, EU) both in the domain of binary prospects and in the subdomain of simple prospects. This systematic approach allows addressing the problems of extending models from a smaller domain to a larger one, comparing the strength of axioms and representations for individual cases, or identifying models in the domain of simple prospects. An additional advantage is the identification of one clean key axiom for each representation while maintaining similar regularity conditions.

Article structure The main characterization results are given in section 2. Section 3 contains an in-depth formal analysis of the relations between our axiom of distributivity, the key axiom of the (1) model, and bisymmetry-like axioms, some of which can be found in the literature. A discussion covering some implications of the main result for model identification, nonbiseparable extensions of the biseparable model, as well as the advantages of the "Aczél (1947)" approach based on certainty equivalents and axioms with the restricted domain is given in Section 4. Section 5 concludes.

2 Main characterization results

A binary prospect is a probability measure on a real interval X with a support consisting of at most two elements. Those elements are called payoffs of a prospect. We let $(x, y; p)$ denote a binary prospect that pays x and y, with probabilities p and $1-p$, respectively. Note that

$$(x, y; p) = (y, x; 1-p) \quad \text{for} \quad x, y \in \mathbb{R},\ p \in [0,1]. \tag{2}$$

If $x = y$ or $p \in \{0, 1\}$ then the prospect $(x, y; p)$ is called degenerate. Any such prospect is identified with its payoff. Consequently, $(x, x; p) = x$ holds for all $x \in X$, $p \in [0, 1]$, and $(x, y; 1) = x$, $(x, y; 0) = y$ hold for all $x, y \in X$. Let $\Delta(X)$ denote the set of all binary prospects. We will also consider families of simple prospects, i.e. binary prospects in which one payoff is set at a certain (known) level. Given $y_0 \in X$, we denote by $\Delta_{y_0}(X)$ the set of simple prospects of the form $(x, y_0; p)$. Whenever we consider simple prospects, we assume that y_0 is known and constant and we write simple prospects as (x, p).

We study the functionals $F : \Delta(X) \to X$ having the representation (1) for $(x, y; p) \in \Delta(X)$. From now on, a utility function is a strictly increasing and continuous function $u : X \to \mathbb{R}$ and a probability weighting function is a strictly increasing function $w : [0, 1] \to [0, 1]$ satisfying $w(0) = 0$ and $w(1) = 1$. Our basic axiom common to all axiomatizations is reflexivity:

(Ref) $F(x) = x$ *for all degenerate prospects* $x \in \Delta(X)$.

In most of our results, we also impose the following regularity condition.

(CM) *F is strictly increasing and continuous in the probability of the higher payoff.*

2.1 Characterizations for simple prospects

We first consider representation (1) for simple prospects $\Delta_{y_0}(X)$, given $y_0 \in X$. The key axiom is permutability.[2]

(Perm) $F(F(x,p);q) = F(F(x,q);p)$, *for* $x \in X$ *and* $p, q \in (0,1)$.

We formulate characterization results based on this axiom for the case when y_0 is the endpoint of X and separately for the case where y_0 belongs to the interior of X. Proofs of all theorems in this paper are provided in the Appendix.

Theorem 1 (**simple only gains/only losses**) *Assume y_0 is an endpoint of X. A function $F : \Delta_{y_0}(X) \to X$ satisfies* **(Ref)**, **(CM)**, *and* **(Perm)** *if and only if there exist a continuous probability weighting function w and a utility function u satisfying $u(y_0) = 0$ such that*

$$F(x,p) = u^{-1}(w(p)u(x)) \quad \textit{for} \quad x \in X,\ p \in [0,1]. \tag{3}$$

Furthermore, (3) *is satisfied with w replaced by another probability weighting function $\tilde{w}$, and u replaced by another utility function $\tilde{u}$ satisfying $\tilde{u}(y_0) = 0$ if and only if there exist $\alpha, r > 0$ such that*

$$\tilde{w}(p) = w(p)^r \qquad \textit{for} \quad p \in [0,1], \tag{4}$$

$$|\tilde{u}(x)| = \alpha|u(x)|^r \quad \textit{for} \quad x \in X. \tag{5}$$

Theorem 2 (**simple gains and losses**) *Let y_0 belong to the interior of X. A function $F : \Delta_{y_0}(X) \to X$ satisfies* **(Ref)**, **(CM)**, *and* **(Perm)** *if and only if there exist continuous probability weighting functions w_-, w_+, and a utility function u satisfying $u(y_0) = 0$ such that*

$$F(x,p) = \begin{cases} u^{-1}(w_-(p)u(x)) & \textit{for} \quad x < y_0,\ p \in [0,1], \\ u^{-1}(w_+(p)u(x)) & \textit{for} \quad x \geq y_0,\ p \in [0,1]. \end{cases} \tag{6}$$

Furthermore, (6) *is satisfied with w_-, w_+ replaced by another pair of probability weighting functions $\tilde{w}_-, \tilde{w}_+$, and u replaced by another utility function $\tilde{u}$ satisfying $\tilde{u}(y_0) = 0$ if and only if there exist $\alpha, \beta, r_-, r_+ > 0$ such that*

$$\tilde{w}_-(p) = w_-(p)^{r_-} \quad p \in [0,1], \tag{7}$$

$$\tilde{w}_+(p) = w_+(p)^{r_+} \quad p \in [0,1], \tag{8}$$

and

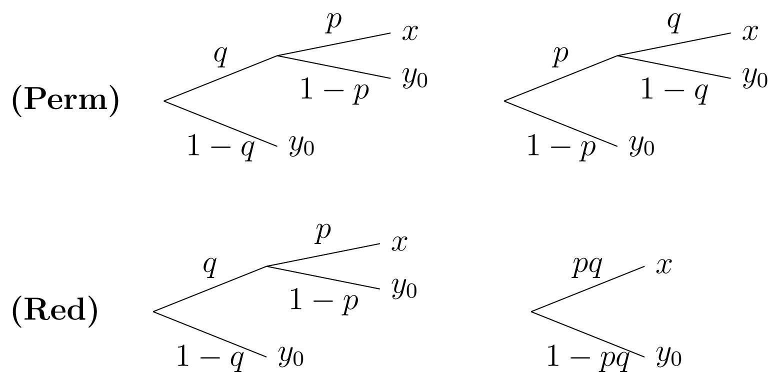
$$\tilde{u}(x) = \begin{cases} -\alpha(-u(x))^{r_-} & \textit{for} \quad x < y_0, \\ \beta u(x)^{r_+} & \textit{for} \quad x \geq y_0. \end{cases} \tag{9}$$

We now characterize the special case of (3) or (6) where the probability weighting function is the identity function. The key axiom in this case is reduction.[3]

[2] The name derives from the fact that the axiom requires the one-parameter set of mappings $y = F(x,p)$ of X in X to be permutable.

[3] In the field of functional equations the term translation equation is used. Moszner (1995) gives a survey of results on its solutions.

Figure 1: Graphical representation of **(Perm)** and **(Red)** where y_0 s fixed.



(Red) $F(F(x,p),q) = F(x,pq)$ *for* $x \in X,\ p,q \in (0,1)$.

Theorem 3 (**simple EU**) *Let* $y_0 \in X$. *A function* $F : \Delta_{y_0}(X) \to X$ *satisfies* **(Ref)**, **(CM)** *and* **(Red)** *if and only if there exists a utility function* u *satisfying* $u(y_0) = 0$ *and:*

$$F(x,p) = u^{-1}(pu(x)) \quad for \quad x \in X,\ p \in [0,1]. \tag{10}$$

Furthermore, (10) *is satisfied with* u *replaced by another utility function* $\tilde{u}$ *satisfying* $\tilde{u}(y_0) = 0$ *if and only if:*

- *in the case where* y_0 *is the endpoint of* X, *there exists* $\alpha > 0$ *such that*

$$\tilde{u}(x) = \alpha u(x) \quad for \ x \in X; \tag{11}$$

- *in the case where* y_0 *is the interior point of* X, *there exist* $\alpha, \beta > 0$ *such that*

$$\tilde{u}(x) = \begin{cases} \alpha u(x) & for \ x < y_0, \\ \beta u(x) & for \ x \geq y_0. \end{cases} \tag{12}$$

The two key axioms of Theorems 1, 2 and 3 are graphically depicted on Figure 1. Each of these graphs shows two equivalent ways of calculating the certainty equivalent of a composite prospect in the form of a tree. **(Perm)** can be explained as follows. Bet Ax pays x if event A occurs and y_0 otherwise. Consider two such bets Ax and Bx, where events A and B are statistically independent. **(Perm)** states that the following two bets are equivalent

b1 pays the certainty equivalent of Ax if B occurs, and y_0 otherwise

b2 pays the certainty equivalent of Bx if A occurs and y_0 otherwise.

(Red) is stronger than **(Perm)** in that it additionally requires that these two bets are equivalent to the bet $(A \cap B)x$. Formally, **(Red)** implies **(Perm)**, which can be seen by applying **(Red)** on both sides of **(Perm)**, but the opposite is not true. For example the representation

$$F(x,p) = xw(p), \quad x \in X,\ p \in [0,1]$$

satisfies **(Perm)** for any probability weighting function w satisfying the conditions of Theorem 2. On the other hand, this representation satisfies **(Red)** if and only if $w(pq) = w(p)w(q)$ for all $p, q \in [0,1]$ which is true if and only if $w(p) = p^{\alpha}$, $p \in [0,1]$ for some $\alpha > 0$.

2.2 Characterization results for binary prospects

We now characterize the model for all binary prospects. The key axiom we apply in this case is distributivity:

(Dist) $F(F(x,y;p),z;q) = F(F(x,z;q),F(y,z;q);p)$ *for* $x \geq y \geq z,\ p,q \in (0,1)$.

Theorem 4 (**binary**) *A function* $F : \Delta(X) \to X$ *satisfies the axioms* **(Ref)**, **(CM)**, *and* **(Dist)** *if and only if there exist a utility function* u *and a continuous probability weighting function* w, *such that* (1) *holds, i.e.*

$$F(x,y;p) = u^{-1}(w(p)u(x) + (1-w(p))u(y)) \quad \textit{for} \quad x \geq y,\ p \in [0,1].$$

Furthermore, (1) *is satisfied with* u *replaced by another utility function* $\tilde{u}$, *and* w *replaced by another probability weighting function* $\tilde{w}$ *if only if* $\tilde{w} = w$ *and there exist* $\alpha, \beta \in \mathbb{R}$, *with* $\alpha > 0$, *such that* $\tilde{u}(x) = \alpha u(x) + \beta$ *holds for all* $x \in X$.

The representation in (1) gives the form of $F(x,y;p)$ for $x \geq y$, i.e. for a given payoff rank. In other cases we use (2) to get

$$F(x,y;p) = u^{-1}((1-w(1-p))u(x) + w(1-p)u(y)) \quad \text{for } x < y,\ p \in [0,1]. \tag{13}$$

The formulas in (1) and (13) differ in general because the probability weighting function w is always applied to the probability of a higher payoff. Thus, the weight assigned to a given payoff depends not only on the probability of that payoff occurring, but also on the rank of the payoff. This is reflected in the **(Dist)** axiom, which holds for ordered payoffs $x \geq y \geq z$. We say that the model is *rank-dependent*. Note, however, that the formulas for $x \geq y$ and $x < y$, given by (1) and (13), would coincide if w was *self-conjugate*, i.e. it satisfied the following condition

$$w(1-p) = 1 - w(p) \quad \text{for} \quad p \in (0,1). \tag{14}$$

We call a biseparable model that meets this condition *rank-independent*. It can be obtained by strengthening the **(Dist)** axiom to hold for any payoffs x, y, z and not just for ordered payoffs.

($\overline{\textbf{Dist}}$) $F(F(x,y;p),z;q) = F(F(x,z;q),F(y,z;q);p)$ *for* $x,y,z \in \mathbb{R},\ p,q \in (0,1)$.

We can derive a rank-independent model as a corollary to Theorem 4. We only assert existence of the representation, because its uniqueness follows directly from Theorem 4.

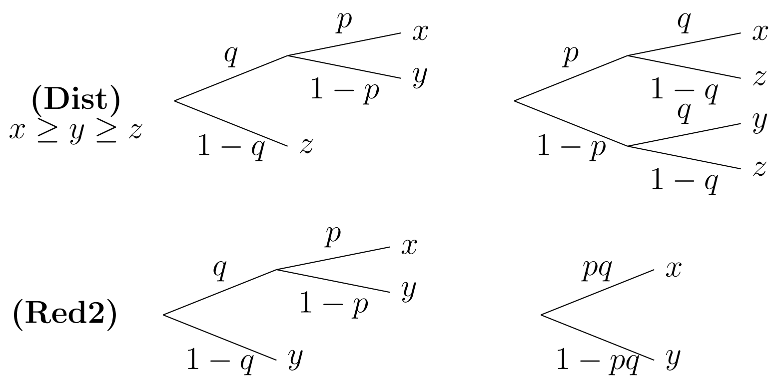
Corollary 1 (**rank-independent**) *A function* $F : \Delta(X) \to X$ *satisfies the axioms* **(Ref)**, **(CM)**, *and* **($\overline{\textbf{Dist}}$)** *if and only if there exist a utility function* u *and a self-conjugate continuous probability weighting function* w, *such that*

$$F(x,y;p) = u^{-1}(w(p)u(x) + (1-w(p))u(y)) \quad \textit{for} \quad (x,y;p) \in \Delta(X). \tag{15}$$

Specializing the model further, we now consider the case where the probability weighting function w is the identity function. The key axiom is now the natural extension of **(Red)**:

(Red2) $F(F(x,y;p),y;q) = F(x,y;pq)$ *for* $x,y \in X,\ p,q \in (0,1)$.

Figure 2: Graphical representation of **(Dist)** and **(Red2)**.



Theorem 5 (**binary EU**) *A function $F : \Delta(X) \to X$ satisfies* **(Ref)**, **(CM)** *and* **(Red2)** *if and only if there exists a utility function u such that*

$$F(x, y; p) = u^{-1}(pu(x) + (1 - p)u(y)) \quad \textit{for} \quad (x, y; p) \in \Delta(X). \tag{16}$$

Furthermore, (16) *is satisfied with u replaced by another utility function $\tilde{u}$ if and only if there are $\alpha, \beta \in \mathbb{R}$, with $\alpha > 0$, such that*

$$\tilde{u}(x) = \alpha u(x) + \beta \quad \textit{for} \quad x \in X. \tag{17}$$

The key axioms of Theorems 4 and 5 are visually represented in Figure 2. It is instructive to compare them with the two key axioms of the corresponding representations for simple prospects. Note that while the second prospect payoff in **(Red)** is fixed at y_0, **(Red2)**, in which the corresponding payoff y is arbitrary, is equivalent to the system of **(Red)** for each y_0. This stronger axiom is sufficient to extend the simple EU to the binary EU model. One might think that an analogous generalization of **(Perm)**, the key axiom of the biseparable model for simple prospects, is sufficient to extend this model from simple to binary prospects. However, this is not the case. Indeed, consider the following natural extension of **(Perm)**:

(Perm2) $F(F(x, y; p), y; q) = F(F(x, y; q), y; p)$ *for* $x \geq y,\ p, q \in (0, 1)$.

For some utility function $\phi : X \to \mathbb{R}$, the model

$$F(x, y; p) = y + \phi^{-1}(p\phi(x - y)), \quad x \geq y,\ p \in [0, 1] \tag{18}$$

satisfies **(Perm2)** but is not biseparable in general. To obtain a biseparable model, one must use the stronger **(Dist)** axiom, of which **(Perm2)** is a special case obtained by setting $z = y$ in the former. It is obvious that **(Perm)** is weaker than both of the above axioms and can be obtained from **(Dist)** by setting $z = y = y_0$.

Even though **(Red)** and **(Perm)** apply on the set of simple, while **(Red2)**, **(Perm2)** and **(Dist)** on the set of binary prospects, it is instructive to compare all of them on the larger set of all binary prospects $\Delta(X)$. The logical relationships between the axioms on this set are depicted below:

$$\begin{array}{ccccc} (\mathbf{Dist}) & \Longrightarrow & (\mathbf{Perm2}) & \Longleftarrow & (\mathbf{Red2}) \\ & & \Downarrow & & \Downarrow \\ & & (\mathbf{Perm}) & \Longleftarrow & (\mathbf{Red}) \end{array}$$

We finish this section with the observation that the regularity axiom **(CM)** assumes continuity and monotonicity only with respect to the probability of the higher payoff. Yet, the representations of Theorems 1–5 imply stronger versions of continuity and monotonicity. In fact, F in all these representations is continuous in each of its variables and monotonic with respect to first-order stochastic dominance (FOSD). For any pair of prospects $(x, y; p)$ and $(x', y', ; p')$, where $x > y$, $x' > y'$ and $p, p' \in (0, 1)$ we say that $(x, y; p)$ dominates $(x', y'; p')$ if the following inequalities hold: $x \geq x'$, $y \geq y'$ and $p \geq p'$, and at least one of these inequalities is strict. We say that F is monotonic with respect to FOSD if $F(x, y; p) > F(x', y'; p')$ holds whenever $(x, y; p)$ dominates $(x', y'; p')$. Monotonicity with respect to FOSD thus holds if F is strictly increasing in payoffs and in the probability of the higher payoff.

3 Distributivity vs. bisymmetry-like axioms

The key axiom of our most general representation (1) is **(Dist)**. This is a new axiom, but it is most closely related to the bisymmetry-like axioms[4], i.e., axioms rooted in the bisymmetry axiom of Aczél (1947) used to characterize the binary quasi-arithmetic mean. Our goal in this section is to provide precise formal relations between the axioms **(Dist)** and $(\overline{\mathbf{Dist}})$ on the one hand, and the corresponding versions of the bisymmetry-like axioms. In doing so, we do not stop at the analysis of the existing axioms, but present an in-depth analysis of several alternative versions of the bisymmetry-like axioms that give the same representation. We start by considering the following functional equation for $p, q \in (0, 1)$

$$F(F(x_1, x_2; p), F(x_3, x_4; p); q) = F(F(x_1, x_3; q), F(x_2, x_4; q); p) \quad \text{for} \quad (x_1, x_2, x_3, x_4) \in X^4. \tag{19}$$

$(\overline{\mathbf{Dist}})$ is a reduced version of (19) supposed to hold for any $p, q \in (0, 1)$. Indeed, setting in (19) $x_1 = x$, $x_2 = y$, $x_3 = x_4 = z$ and applying **(Ref)** yields $(\overline{\mathbf{Dist}})$. For $p = q$ equation (19) coincides with the bisymmetry axiom of Aczél (1947).[5] It turns out that under the appropriate regularity conditions, it yields the same (rank-independent) version of the biseparable model as the $(\overline{\mathbf{Dist}})$ axiom. Likewise, fixing one probability in (19) and letting only one vary, also yields the model with self-conjugate w. More precisely, the following counterpart of Corollary 1 holds.

Theorem 6 *Assume that $F : \Delta(X) \to X$ satisfies* **(Ref)***, is continuous and strictly increasing in each of its payoffs and it is strictly increasing in the probability of the higher payoff.*

[4]Bisymmetry-like axioms is a term used by Köbberling and Wakker (2003, p.395) to refer to axioms such as multisymmetry (Chew and Epstein, 1989; Quiggin, 1982; Pfanzagl, 1959), act-independence (Ghirardato and Marinacci, 2001).

[5]For $p = q \in (0, 1)$, defining $M : X^2 \to X$ by $M(x, y) := F(x, y; p)$ yields the classic bisymmetry equation $M(M(x_1, x_2), M(x_3, x_4)) = M(M(x_1, x_3), M(x_2, x_4))$ for $(x_1, x_2, x_3, x_4) \in X^4$.

Then the following statements are pairwise equivalent:

(i) F *is of the form* (15) *with some utility function* u *and a self-conjugate probability weighting function* w*;*

(ii) F *satisfies* (19) *for any* $p \in (0,1)$ *and* $q = p$*;*

(iii) F *satisfies* (19) *for any* $p \in (0,1)$ *and some* $q \in (0,1)$*;*[6]

(iv) F *satisfies* (19) *for any* $p, q \in (0,1)$.

Remark 1 *Replacing in the statement (iii) "some* $q \in (0,1)$*" with "*$q = 0.5$*" and applying the recent result of Burai et al. (2021) we can drop the continuity assumption in Theorem 6.*

Remark 2 *If* F *in Theorem 6 additionally satisfies* ***(CM)****, then* w *acting in (i) is continuous and so, in view of Corollary 1, each of the conditions (i)–(iv) is equivalent to* $(\overline{\textbf{Dist}})$.

Theorem 6 shows that (19), even assumed to hold with natural restrictions in the *domain of probabilities*, always yields the rank-independent version of the biseparable model. We will now focus on restrictions of (19) in the *domain of payoffs* and identify those that yield a more general biseparable model. The following result characterizes the anticipated utility model for binary prospects, in which the probability weighting function w satisfies $w(0.5) = 0.5$, but is not necessarily self-conjugate.

Theorem 7 *Assume that* $F : \Delta(X) \to X$ *satisfies* **(Ref)** *and monotonicity with respect to each of its payoffs. Then the following statements are equivalent:*

(i) F *is of the form* (1) *with some utility function* u *and a probability weighting function* w *satisfying* $w(0.5) = 0.5$*;*

(ii) F *satisfies* (19) *with* $p = q = 0.5$*, and*

$$F(F(x_1, x_2; p), F(x_3, x_4; p); 0.5) = F(F(x_1, x_3; 0.5), F(x_2, x_4; 0.5); p) \tag{20}$$

for all $p \in (0,1)$ *and* $(x_1, x_2, x_3, x_4) \in X^4$ *satisfying* $x_1 \geq x_2$ *and* $x_3 \geq x_4$.

Condition (ii) is closely related to the independence axiom of Quiggin and Wakker (1994), the key axiom in their derivation of the anticipated utility model. We highlight the differences. Our axiom uses certainty equivalents and applies to binary gambles. We explicitly state that for $p = q = 0.5$ (19) holds without payoff restrictions. In Quiggin and Wakker (1994) this is also the case, but it is hidden in their notation for equal chance binary gambles. This is what allows us to significantly shorten the proof by using the classic results based on the bisymmetry axiom of Aczél (1947) that do not assume payoff restrictions. In fact, by using the result of Burai et al. (2021) we can derive the representation for equal chance binary prospects and then extend it for all other binary prospects using (20) without assuming continuity.

[6] Note that since the roles of p and q in (19) are symmetric, we can fix q without loss of generality.

Finally, we are interested in relaxing (19) by imposing payoff restrictions that would yield the general biseparable model (1). In what follows, given $A \subset X^4$, we shall say that $F : \Delta(X) \to X$ satisfies the bisymmetry equation on A provided

$$F(F(x_1, x_2; p), F(x_3, x_4; p); q) = F(F(x_1, x_3; q), F(x_2, x_4; q); p) \tag{21}$$

holds for any $p, q \in (0, 1)$ and $(x_1, x_2, x_3, x_4) \in A$. Let Σ be a family of all permutations on $\{1, 2, 3, 4\}$. For any $\sigma \in \Sigma$ define a rank-ordered payoff space

$$A_\sigma = \{(x_1, x_2, x_3, x_4) \in X^4 : \; x_{\sigma(1)} \geq x_{\sigma(2)} \geq x_{\sigma(3)} \geq x_{\sigma(4)}\}.$$

There are 4! possible rank-ordered payoff spaces corresponding to the elements of Σ. We partition them into the following three sets:

$$\begin{aligned}
\Sigma_1 :=& \{\sigma \in \Sigma : (\sigma(1) - \sigma(2))(\sigma(3) - \sigma(4)) > 0 \text{ and } (\sigma(1) - \sigma(3))(\sigma(2) - \sigma(4)) > 0\},\\
\Sigma_2 :=& \{\sigma \in \Sigma : (\sigma(1) - \sigma(2))(\sigma(3) - \sigma(4)) < 0 \text{ and } (\sigma(1) - \sigma(4))(\sigma(2) - \sigma(3)) > 0\},\\
\Sigma_3 :=& \{\sigma \in \Sigma : (\sigma(1) - \sigma(4))(\sigma(2) - \sigma(3)) < 0 \text{ and } (\sigma(1) - \sigma(3))(\sigma(2) - \sigma(4)) < 0\}.
\end{aligned}$$

We now show that due to the symmetries of the equation (21) combined with (2), if F satisfies the bisymmetry equation on $A_{\sigma'}$ for some $\sigma' \in \Sigma_i$, $i \in \{1, 2, 3\}$ then it satisfies the bisymmetry equation on A_σ for all $\sigma \in \Sigma_i$.

Lemma 1 *Let $i \in \{1, 2, 3\}$. Then, for every $\sigma \in \Sigma_i$, we have*

$$\Sigma_1 \circ \sigma := \{\pi \circ \sigma : \pi \in \Sigma_1\} = \Sigma_i. \tag{22}$$

Lemma 2 *Let $F : \Delta(X) \to X$ and $i \in \{1, 2, 3\}$. Furthermore, let*

$$\Sigma_i(F) := \{\sigma \in \Sigma_i : \; F \text{ satisfies the bisymmetry equation on } A_\sigma\}.$$

Then either $\Sigma_i(F) = \emptyset$ or $\Sigma_i(F) = \Sigma_i$.

Equipped with Lemma 2, we now show a formal connection between **(Dist)** and **($\overline{\textbf{Dist}}$)** on the one hand, and the bisymmetry equation on different rank-ordered spaces on the other.

Theorem 8 *Assume that $F : \Delta(X) \to X$ satisfies* **(Ref)** *and* **(CM)**. *Then the following statements are pairwise equivalent:*

(i) F is of the form (15) *with some utility function u and a self-conjugate continuous probability weighting function w;*

(ii) F satisfies **($\overline{\textbf{Dist}}$)**;

(iii) F satisfies the bisymmetry equation on A_σ for some $\sigma \in \Sigma_2$;

(iv) F satisfies the bisymmetry equation on A_σ for every $\sigma \in \Sigma_2$;

(v) F satisfies the bisymmetry equation on X^4.

Theorem 9 *Assume that $F : \Delta(X) \to X$ satisfies* **(Ref)** *and* **(CM)**. *Then the following statements are pairwise equivalent:*

(i) F is of the form (1) *with some utility function u and a continuous probability weighting function w;*

(ii) F satisfies **(Dist)**;

(iii) F satisfies the bisymmetry equation on A_σ for some $\sigma \in \Sigma_1$;

(iv) F satisfies the bisymmetry equation on A_σ for every $\sigma \in \Sigma_1$;

(v) F satisfies the bisymmetry equation on

$$\mathcal{A} := \{(x_1, x_2, x_3, x_4) \in X^4 : (x_1 - x_2)(x_3 - x_4) \geq 0 \text{ and } (x_1 - x_3)(x_2 - x_4) \geq 0\}.$$

In light of our previous results on the characterization of the general biseparable model and its rank-independent version using these first axioms, we thus obtain additional characterization results using the bisymmetric equation. Furthermore, the above theorems emphasize the special role of the orderings in Σ_1 in obtaining the general bisymmetric representation and the orderings in Σ_2 in obtaining its rank-independent version with w being self-conjugate. In particular, F satisfies the bisymmetry equation on X^4 whenever it satisfies the bisymmetry equation on the payoff space ordered by just one $\sigma \in \Sigma_2$. This result strengthens Lemma 2. On the other hand, if F satisfies the bisymmetry equation in the payoff space ordered by $\sigma \in \Sigma_1$, then F satisfies the bisymmetry equation on the set $\mathcal{A}$, whose definition, stated in terms of payoffs rather than payoff orders, simplifies the intuitive interpretation of condition (v) of Theorem 9.

In fact, consider the prospect $(x, y; p)$. As is evident from the comparison of (1) and (13), in the general biseparable model, the weight applied to the payoff x depends on its rank relative to y: it is either $w(p)$ if $x \geq y$, or $1 - w(1 - p)$ if $x < y$. If payoff ranks in (21) are unrestricted, it may thus happen that the same payoff is multiplied by $w(p)$ on the left-hand side and by $1 - w(1 - p)$ on the right-hand side of the equation. It forces w to satisfy the self-conjugate condition $w(p) = 1 - w(1 - p)$. The payoff restrictions in $\mathcal{A}$ prohibit this from happening. Consider all p-probability prospects in (21). If payoff pairs (x_1, x_2) and (x_3, x_4), appearing on the left-hand side of (21), are ordered the same, which is the first condition in the definition of $\mathcal{A}$, then the payoff pair $(F(x_1, x_3; q), F(x_2, x_4; q))$, which appears on the right-hand of (21), is also ordered the same. Similarly, if payoff pairs (x_1, x_3) and (x_2, x_4), appearing on the right-hand side of (21), are ordered the same, which is the second condition in the definition of $\mathcal{A}$, then the payoff pair $(F(x_1, x_2; p), F(x_3, x_4; p))$, which appears on the right-hand of (21), is also ordered the same.

4 Discussion

We discuss implications of our results for simple prospects for model identification. Then we illustrate the issue of nonbiseparable extensions of the biseparable model. Next, we illustrate the advantage of our axioms for binary prospects as compared to axioms deduced as a special case of the model for general prospects. Finally, we show that the biseparable model allows for preference reversals while the binary rank-dependent utility model does not.

4.1 Identifiability of the models for simple prospects

The uniqueness part of Theorem 2 shows that model (6) is only partially identified on the set of simple prospects. It implies in particular that by taking the positive powers of the utility function and the probability weighting function separately for the gain and loss parts, we obtain an equivalent representation. This has important consequences in modeling individual attitudes towards risk. A typical shape for a probability weighting function is an inverted S-shaped function (Wakker, 2010, p.204), in which (winning) probabilities below a certain threshold are overweighted and probabilities above the threshold are underweighted. The location of the threshold, which is the interior fixed point of the probability weighting function, is an important element in modeling individual attitudes towards risk. Theorem 2 implies that for simple prospects, which many empirical studies use only (Preston and Baratta, 1948) or mostly (Tversky and Kahneman, 1992; Gonzalez and Wu, 1999), the threshold cannot be correctly identified. To illustrate this, let's fix $y_0 \in X$ and assume that:

$$F(x,p) = u^{-1}(w(p)u(x)), \quad x \geq y_0,\ p \in [0,1],$$

for some utility function u and probability weighting function w. Consider any point $p_0 \in (0,1)$ that is not a fixed point of w. Since $w(p_0) \in (0,1)$, for any such point one can find $r > 0$ such that $w(p_0) = p_0^r$. Let's define a new pair of functions $\tilde{w}, \tilde{u}$ by $\tilde{w}(p) = w(p)^{1/r}$ and $\tilde{u}(x) = u(x)^{1/r}$, which, according to Theorem 2, generate an equivalent representation. Note that although p_0 is not a fixed point of w, it is a fixed point of $\tilde{w}$:

$$\tilde{w}(p_0) = w(p_0)^{1/r} = (p_0^r)^{1/r} = p_0.$$

In this way, an equivalent representation can be constructed in which the probability weighting function has a fixed point at any point $(0,1)$. This observation is visually illustrated in Figure 3, which shows several equivalent pairs of utility functions and probability weighting functions, each of the latter with a different interior fixed point.

One can think of y_0 in simple prospects as a reference point, relative to which other payoffs are evaluated with payoffs above y_0 called gains and payoffs below y_0 called losses. Theorem 2 implies that the ratio of utilities for gains and losses (i.e loss aversion) is unidentified and hence the utility scale consists of two separate scales, one for gains and one for losses. Simple prospects are insufficient to identify loss aversion, because a simple prospect is either a gain prospect or a loss prospect, but never a mixed prospect.

In reference dependent models, where loss aversion plays an important role, it is often assumed that the utility function (sometimes also called the value function) takes the following form:

$$u(x) = \begin{cases} (x - y_0)^{\alpha}, & x \geq y_0, \\ -\lambda(y_0 - x)^{\beta}, & x < y_0. \end{cases}$$

(a) $\tilde{u}(x) = u(x)^r$, $x \geq y_0 = 0$ (b) $\tilde{w}_+(p) = w_+(p)^r$, $r > 0$

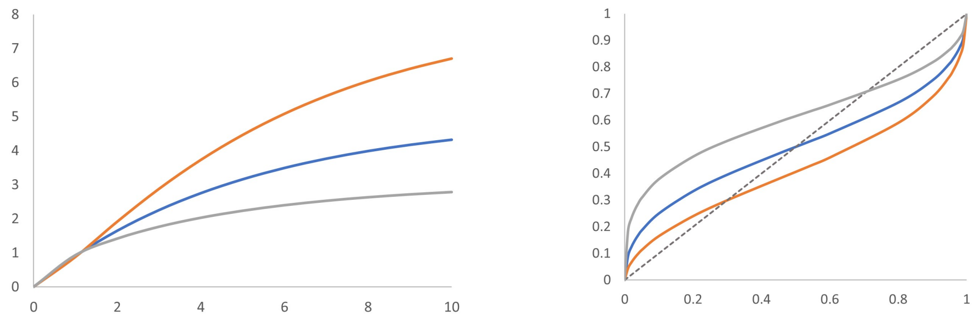


Figure 3: Three pairs of $\tilde{u}, \tilde{w_+}$, each yielding the same $F(x,p)$.

for $\alpha, \beta > 0$, where $\lambda > 0$ is a loss aversion parameter. Suppose that $F(x,p)$ is given by (2) with utility function u of the above form and continuous probability weighting functions for gains and losses given by w_+, w_-, respectively. Theorem 2 implies that there is an equivalent representation of $F(x,p)$ in which the utility function is piece-wise linear. Indeed, define $\tilde{u}(x) = u(x)^{1/\alpha}$ for $x \geq y_0$, $\tilde{u}(x) = -(-u(x))^{1/\beta}$ for $x < y_0$, and $\tilde{w}_+(p) = w_+(p)^{1/\alpha}$, $\tilde{w}_-(p) = w_-(p)^{1/\beta}$ for $p \in [0,1]$. This generates an equivalent representation of $F(x,p)$ where $\tilde{u}$ is piecewise linear and the new loss aversion parameter is $\lambda' = \lambda^{1/\beta}$:

$$\tilde{u}(x) = \begin{cases} x - y_0, & x \geq y_0 \\ \lambda'(x - y_0), & x < y_0. \end{cases}$$

4.2 Nonbiseparable extensions of the biseparable model

The problem of extending a given model from a smaller domain to a larger one can be studied at the level of axioms or at the level of representation. Let us consider the problem of extending the biseparable model for simple prospects to the biseparable model for binary prospects at the representation level. Fix $y_0 \in X$ and consider the (6) model for prospects $(x,p) \in \Delta_{y_0}(X)$ such that $x \geq y_0$. This model is a special case of the (1) model. Indeed, set $y = y_0$ in (1) and replace u with $\tilde{u} = u - u(y_0)$, which immediately gives $F(x,p) = \tilde{u}^{-1}(w(p)\tilde{u}(x))$ for $x \geq y_0$.

It shows that (1) is one possible extension of (6). However, there are other extensions as well, for example:

$$F(x,y;p) = y + \phi^{-1}(w(p)\phi(x-y)), \quad x \geq y, \ p \in [0,1]. \tag{23}$$

Note that (23) coincides with the simple general model (6) on the set of simple prospects. In fact, taking $y_0 \in X$ and defining the utility function by $u(x) = \phi(x - y_0)$ for $x \in X$, we get $F(x, y_0; p) = u^{-1}(w(p)u(x))$ for $x \geq y_0$, $p \in [0,1]$. However, (23) differs from the general binary model (1). uzasadnić, że (23) nie jest biseparowalny na binary, ale jest na simple.

It can be obtained from (6) for any y_0 by assuming $F(x,y;p)$ is translative. Indeed, let $F(x,y_0;p) = u^{-1}(w(p)u(x))$, for some probability weighting function w and utility function u satisfying $u(y_0) = 0$. Define another utility function by $\phi(x) = u(x + y_0)$, $x \in X$. This function satisfies $\phi(0) = 0$. Then for $x \geq y$, using translativity, we get

$$\begin{aligned} F(x, y; p) &= y - y_0 + F(x - y + y_0, y_0; p) \\ &= y - y_0 + u^{-1}(w(p)u(x - y + y_0)) \\ &= y + \phi^{-1}(w(p)\phi(x - y)). \end{aligned}$$

The conclusion is that the CE models that coincide with (6) on the set of simple prospects are strictly more general than the CE models that coincide with (1) on the set of binary prospects. This clearly has implications for model testing.

4.3 The use of the domain-specific axioms

The important property of all our axioms is that they are defined one the same domain as the representation they yield. This is not true in many other axiomatizations. For example (Ghirardato and Marinacci, 2001, Theorem 3) characterize a biseparable representation under uncertainty, in which the domain of their key axiom (Weak Certainty Independence) goes beyond the set of binary acts.

We illustrate this property by comparing our reduction axiom for binary prospects with the quasilinearity axiom of de Finetti (Hardy et al., 1934, p. 157-163), used as the key axiom to characterize the quasilinear mean. The quasilinearity axiom is stated for mean values (denoted by F) of finite distribution functions $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ on a bounded real interval $[a, b]$ and their probability mixtures:[7]

(QL) *If* $F(\mathbf{X}) = F(\mathbf{Y})$, *then* $F(\mathbf{X}, \mathbf{Z}; q) = F(\mathbf{Y}, \mathbf{Z}; q)$ *for all* $\mathbf{Z}$ *and* $q \in (0, 1)$.

Suppose we want to obtain a version of **(QL)** in the domain of binary prospects. In order to do it, it does not suffice to restrict $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ to be binary. However, applying **(QL)** for $\mathbf{X} = (x, y; p)$, $\mathbf{Y} = (x', y; p')$ and $\mathbf{Z} = y$ where $x, x', y \in X$, $p, p' \in [0, 1]$, we obtain

$$\text{If } F(x, y; p) = F(x', y; p'), \text{ then } F(x, y; pq) = F(x', y; p'q) \quad \text{for} \quad q \in (0, 1). \tag{24}$$

Hence, in the domain of binary prospects, **(QL)** implies (24). Note that if **(Ref)** is true, then **(Red2)** is equivalent to the restricted version of (24) in which $p' = 1$, i.e.

$$\text{If } F(x, y; p) = x' \text{ then } F(x, y; pq) = F(x', y; q) \quad \text{for} \quad q \in (0, 1). \tag{25}$$

The above argument shows that the version of **(QL)** for binary prospects is still stronger, given **(Ref)**, than our domain-specific axiom **(Red2)**. This is an illustration of the general property that axioms tailored to a given subdomain of prospects (binary or simple prospects) are more efficient and minimal than axioms obtained by formulating the axiom for the general domain in the given subdomain.

[7] Note that this axiom can be viewed as the analogue of the independence (also called substitution) axiom for preferences stated in terms of CEs.

4.4 Certainty Equivalents vs. preferences

The primitive in our approach is the certainty equivalent. Most often, however, the preference relation is considered primitive. If this relation is monotonic (the more certain money the better), transitive, and there is a certainty equivalent for each prospect, then the certainty equivalent functional orders prospects in the same way as preferences. However, there are well-defined preferences that are not representable by any function. Such preferences exhibit preference reversal, where one prospect is preferred over the other in direct choice but has a lower certainty equivalent (Lichtenstein and Slovic, 1971; Grether and Plott, 1979; Seidl, 2002). Consider the following preference[8] relation $\succcurlyeq \subset \Delta_0(X) \times \Delta_0(X)$

$$(x,p) \succcurlyeq (y,q) \iff pw^{-1}\left(\frac{u(x)}{u(\max(x,y))}\right) \geq qw^{-1}\left(\frac{u(y)}{u(\max(x,y))}\right), \tag{26}$$

where $u : \mathbb{R} \to \mathbb{R}$ and $w : [0,1] \to [0,1]$ are strictly increasing and invertible functions satisfying $u(0) = 0$ and $w(0) = 0$, $w(1) = 1$. This model yields CE of the form (3).

$$(F(x,p),1) \sim (x,p) \iff w^{-1}\left(\frac{u(F(x,p))}{u(x)}\right) = p \iff F(x,p) = u^{-1}(w(p)u(x)).$$

Thus the CE yields the following order over prospects:

$$F(x,p) \geq F(y,q) \iff \frac{u(x)}{u(y)} \geq \frac{w(q)}{w(p)}.$$

On the other hand, assuming that $0 < x < y$ and $0 < q < p < 1$, we get

$$(x,p) \succcurlyeq (y,q) \iff pw^{-1}\left(\frac{u(x)}{u(y)}\right) \geq q \iff \frac{u(x)}{u(y)} \geq w\left(\frac{q}{p}\right).$$

Therefore, unless w is a power function, in which case $w\left(\frac{q}{p}\right) = \frac{w(q)}{w(p)}$ for each $0 < q < p < 1$, these two orderings differ.

The above argument shows that, assuming that CEs exist, the class of preferences generating CEs of the form (1) is noticeably more general than the preferences in the binary rank-dependent utility model. This further justifies our focus on the (1) model.

5 Conclusions

In this article, we characterized certainty equivalents of the form (1) for simple and binary prospects. The results help to understand the limitations of the popular method of eliciting preferences for simple or binary prospects and extrapolating the results to more complex prospects, either for descriptive or decision-support purposes. Additionally, our results on the uniqueness of the representations for simple and binary prospects provide us with guidance for testing and identifying models on various datasets. These results may be helpful in designing experiments aimed at eliciting individual attitudes towards risk.

Future research will focus on providing analogous characterization results: a) in the domain of ambiguity/uncertainty in which objective probabilities are unknown to the decision

[8] This is a special case of Range Utility Theory (Baucells et al., 2024) for simple prospects in $\Delta_0(X)$.

maker, b) for preferences instead of certainty equivalents c) for prospects with more than two payouts. Such characterizations should complement knowledge about how much stronger the conditions should be to extend a given representation for a given domain to a larger domain.

A Proofs

We start with four lemmas that will be used in the proofs of Theorems 1–4.

Lemma 3 *Let I be a real interval and $y_0 \in I$ be its endpoint. Assume that $u_1, u_2 : I \to \mathbb{R}$ are utility functions with $u_1(y_0) = u_2(y_0) = 0$ and w_1, w_2 are continuous probability weighting functions. Then*

$$u_1^{-1}(w_1(p)u_1(x)) = u_2^{-1}(w_2(p)u_2(x)) \quad \textit{for} \quad x \in I,\ p \in [0,1], \tag{27}$$

if and only if there exist $\alpha, r \in (0,\infty)$ such that

$$w_2(p) = w_1(p)^r \qquad \textit{for} \quad p \in [0,1], \tag{28}$$

$$|u_2(x)| = \alpha |u_1(x)|^r \quad \textit{for} \quad x \in I. \tag{29}$$

Proof. Standard computations show that (28) and (29) imply (27). For the converse part, assume that (27) holds. First consider the case when $y_0 = \min I$. Replacing in (27) x by $u_1^{-1}(x)$ and putting

$$f := u_2 \circ u_1^{-1}, \tag{30}$$

we obtain

$$f(w_1(p)x) = w_2(p)f(x) \quad \text{for} \quad x \in u_1(I),\ p \in [0,1].$$

Setting $x = x_0 \in u_1(I) \setminus \{0\}$ gives

$$w_2(p) = \frac{f(w_1(p)x_0)}{f(x_0)} \quad \text{for} \quad p \in [0,1] \tag{31}$$

and so plugging it back yields

$$f(w_1(p)x) = \frac{f(w_1(p)x_0)}{f(x_0)} f(x) \quad \text{for} \quad x \in u_1(I),\ p \in [0,1].$$

Moreover, as w_1 is a continuous probability weighting function, we have $w_1([0,1]) = [0,1]$. Thus we get the following Pexider equation on a restricted domain

$$f(xy) = \frac{f(yx_0)}{f(x_0)} f(x) \quad \text{for} \quad (x,y) \in u_1(I) \times [0,1]. \tag{32}$$

Note that as u_1 is a utility function and $u_1(y_0) = 0$, $u_1(I)$ is a real interval having 0 as its left endpoint. Thus the interior of the domain is an open rectangle contained in $(0,\infty)^2$. Hence, according to (Sobek, 2010, Corollary 2) the solutions of (32) can be uniquely extended to the solutions of the corresponding Pexider equation on $(0,\infty)^2$. So, as f is strictly increasing and continuous with $f(0) = 0$, using the standard results (see for example Theorem 13.1.6 of Kuczma, 2008), we conclude that there exist $\alpha, r \in (0, \infty)$ such that

$$f(x) = \alpha x^r \quad \text{for} \quad x \in u_1(I)$$

and

$$\frac{f(yx_0)}{f(x_0)} = y^r \quad \text{for} \quad y \in [0, 1].$$

Hence, in view of (30) and (31), we obtain (29) and (28), respectively, which completes the proof for the case when $y_0 = \min I$.

We now assume that $y_0 = \max I$. Let $\tilde{I} := \{2y_0 - x : x \in I\}$ and $\tilde{u}_i : \tilde{I} \to \mathbb{R}$ for $i \in \{1, 2\}$ be given by

$$\tilde{u}_i(x) = -u_i(2y_0 - x) \quad \text{for} \quad x \in \tilde{I}. \tag{33}$$

Then $y_0 = \min \tilde{I}$ and, for $i \in \{1, 2\}$, $\tilde{u}_i$ is a utility function with $\tilde{u}_i(y_0) = 0$. Moreover, in view of (27) and (33), we have

$$\tilde{u}_1^{-1}(w_1(p)\tilde{u}_1(x)) = \tilde{u}_2^{-1}(w_2(p)\tilde{u}_2(x)) \quad \text{for} \quad x \in \tilde{I},\ p \in [0, 1].$$

Therefore, applying the already proved part, we conclude that there exist $\alpha, r \in (0, \infty)$ such that (28) holds and $\tilde{u}_2(x) = \alpha \tilde{u}_1(x)^r$ for $x \in \tilde{I}$. Hence, taking (33) into account, we get (29) and the proof is completed. □

Lemma 4 *Let $X = [y_0, a)$ for some $a \in (y_0, \infty)$. Assume that for every $b \in (y_0, a)$ there exist a continuous probability weighting function w_b and a utility function $u_b : [y_0, b] \to \mathbb{R}$ such that $u_b(y_0) = 0$ and*

$$F(x, p) = u_b^{-1}(w_b(p)u_b(x)) \quad \textit{for} \quad x \in [y_0, b],\ p \in [0, 1]. \tag{34}$$

Then $w_z = w_{z'} =: w$ for $z, z' \in (y_0, a)$ and there exists a utility function $u : X \to \mathbb{R}$, with $u(y_0) = 0$, such that (3) holds.

Proof. Let $(a_n : n \in \mathbb{N})$ be a strictly increasing sequence of elements of X such that $\lim_{n\to\infty} a_n = a$. Then, according to (34), for every $n \in \mathbb{N}$, we have

$$u_{a_n}^{-1}(w_{a_n}(p)u_{a_n}(x)) = u_{a_1}^{-1}(w_{a_1}(p)u_{a_1}(x)) \quad \text{for} \quad x \in [y_0, a_1],\ p \in [0, 1].$$

Hence, applying Lemma 3, we obtain that for every $n \in \mathbb{N}$ there exist $c_n, r_n \in (0, \infty)$ such that

$$u_{a_n}(x) = c_n u_{a_1}(x)^{r_n} \quad \text{for} \quad x \in [y_0, a_1],\ n \in \mathbb{N} \tag{35}$$

and

$$w_{a_n}(p) = w_1(p)^{r_n} \quad \text{for} \quad p \in [0, 1],\ n \in \mathbb{N}. \tag{36}$$

From (34) we derive that

$$u_{a_n}^{-1}(w_{a_n}(p)u_{a_n}(a_n)) = u_{a_{n+1}}^{-1}(w_{a_{n+1}}(p)u_{a_{n+1}}(a_{n+1})) \quad \text{for} \quad n \in \mathbb{N},\ p \in [0, 1].$$

Hence, in view of (36), we get

$$u_{a_n}^{-1}(w_1(p)^{r_n}u_{a_n}(a_n)) = u_{a_{n+1}}^{-1}(w_1(p)^{r_{n+1}}u_{a_{n+1}}(a_{n+1})) \quad \text{for} \quad n \in \mathbb{N},\ p \in [0, 1].$$

So, taking $p_n \in (0,1]$ such that $w_1(p_n) = \left(\frac{u_{a_n}(a_1)}{u_{a_n}(a_n)}\right)^{\frac{1}{r_n}}$, for any $n \in \mathbb{N}$ we obtain

$$\left(\frac{u_{a_n}(a_n)}{u_{a_n}(a_1)}\right)^{\frac{1}{r_n}} = \left(\frac{u_{a_{n+1}}(a_n)}{u_{a_{n+1}}(a_1)}\right)^{\frac{1}{r_{n+1}}}.$$

Thus, in view of (35), we get

$$\left(\frac{u_{a_n}(a_n)}{c_n}\right)^{\frac{1}{r_n}} = \left(\frac{u_{a_{n+1}}(a_n)}{c_{n+1}}\right)^{\frac{1}{r_{n+1}}} \quad \text{for} \quad n \in \mathbb{N}. \tag{37}$$

Define a function $u : X \to \mathbb{R}$ in the following way

$$u(x) = u_{a_1}(x) \quad \text{for} \quad x \in [y_0, a_1], \tag{38}$$

$$u(x) = \left(\frac{u_{a_{n+1}}(x)}{c_{n+1}}\right)^{\frac{1}{r_{n+1}}} \quad \text{for} \quad x \in (a_n, a_{n+1}],\ n \in \mathbb{N}. \tag{39}$$

From (37)-(39) we derive that, for every $n \in \mathbb{N}$, u is continuous on $[a_n, a_{n+1}]$ and so, it is continuous. Moreover u, being strictly increasing on $[a_n, a_{n+1}]$ for $n \in \mathbb{N}$, is strictly increasing. It follows from (38) that $u(y_0) = u_1(y_0) = 0$. Finally, taking $x \in (y_0, a)$ and setting $m := \min\{n \in \mathbb{N} : x \le a_n\}$, in view of (34)-(36), for any $p \in [0,1]$, we obtain

$$F(x,p) = u_{b_n}^{-1}(w_{b_n}(p)u_{a_n}(x)) = u_{a_n}^{-1}(w_1(p)^{r_n} c_n u(x)^{r_n}) = u^{-1}(w_1(p)u(x)).$$

Taking **(Ref)** into account, we have also

$$F(y_0, p) = F(y_0) = y_0 = u^{-1}(0) = u^{-1}(w_1(p)u(y_0)).$$

In this way we have proved that (3) holds with $w := w_1$, which completes the proof. □

Lemma 5 *Assume that I is a real interval, $u_1, u_2 : I \to \mathbb{R}$ are utility functions and $\gamma, \theta \in (0,1)$. Then*

$$u_1^{-1}(\gamma u_1(x) + (1-\gamma)u_1(y)) = u_2^{-1}(\theta u_2(x) + (1-\theta)u_2(y)) \quad \textit{for} \quad x, y \in I,\ x \ge y. \tag{40}$$

if and only if $\gamma = \theta$ and there exist $\alpha \in (0,\infty)$ and $\beta \in \mathbb{R}$ such that

$$u_2(x) = \alpha u_1(x) + \beta \quad \textit{for} \quad x \in I. \tag{41}$$

Proof. The 'if' part is standard. We now prove the 'only if' part. Assume that (40) holds and let f be given by (30). Then, as u_1 and u_2 are utility functions, f is strictly increasing and continuous. Furthermore, replacing in (40) x and y by $u_1^{-1}(x)$ and $u_1^{-1}(y)$, respectively, we get

$$f(\gamma x + (1-\gamma)y) = \theta f(x) + (1-\theta)f(y) \quad \text{for} \quad x, y \in u_1(I),\ x \ge y. \tag{42}$$

Let

$$D := \{(\gamma s, (1-\gamma)t) :\ s, t \in u_1(I),\ s \ge t\}.$$

Then, taking $(x,y) \in D$, we have $x = \gamma s$ and $y = (1-\gamma)t$ for some $s,t \in u_1(I)$ with $s \geq t$, and so applying (42) we obtain

$$f(x+y) = f(\gamma s + (1-\gamma)t) = \theta f(s) + (1-\theta)f(t) = \theta f\left(\frac{x}{\gamma}\right) + (1-\theta) f\left(\frac{y}{1-\gamma}\right).$$

Hence, taking $D_1 := \{\gamma s : s \in u_1(I)\}$ and $D_2 := \{(1-\gamma)t : t \in u_1(I)\}$, we get

$$f(x+y) = g(x) + h(y) \quad \text{for} \quad (x,y) \in D,$$

where $g : D_1 \to \mathbb{R}$ and $h : D_2 \to \mathbb{R}$ are given by $g(x) = \theta f\left(\frac{x}{\gamma}\right)$ for $x \in D_1$ and $h(y) = (1-\theta) f\left(\frac{y}{\gamma}\right)$ for $y \in D_2$, respectively. Note that the above equation is a Pexider equation on a restricted domain D. Moreover, as $u_1(I)$ is an interval, D is a connected subset of $\mathbb{R}^2$ with a nonempty interior. Furthermore

$$D_+ := \{x + y : (x,y) \in D\} = \{\gamma s + (1-\gamma)t,\ s,t \in u_1(I),\ s \geq t\} = u_1(I).$$

Therefore, applying the extension result of (Radó and Baker, 1987, Corollary 3), we obtain that there exists an additive mapping $a : \mathbb{R} \to \mathbb{R}$ and a $\beta \in \mathbb{R}$ such that $f(x) = a(x) + \beta$ for every x belonging to the interior of $u_1(I)$. Since f is continuous and strictly increasing, so is a, and hence, applying the standard argument (cf. e.g. Kuczma, 2008, Theorem 5.5.2), we get

$$f(x) = \alpha x + \beta \quad \text{for} \quad x \in u_1(I),$$

with some $\alpha \in (0,\infty)$. Thus, making use of (30), we obtain (41). Furthermore, inserting f into (42) gives

$$(\theta - \gamma)(x - y) = 0 \quad \text{for} \quad x,y \in u_1(I),\ x \geq y,$$

which yields $\theta = \gamma$ and completes the proof. □

Lemma 6 *Assume that for any z in the interior of X there exist a utility function $u_z : X_{\geq z} \to \mathbb{R}$ and a continuous probability weighting function w_z such that*

$$F(x,y;p) = u_z^{-1}\left(w_z(p)u_z(x) + (1-w_z(p))u_z(y)\right) \quad \textit{for} \quad x \geq y \geq z,\ p \in [0,1]. \tag{43}$$

Then $w_z = w_{z'} =: w$ for any z and z' in the interior of X and there exists a utility function $u : X \to \mathbb{R}$ such that (1) holds.

Proof. In view of (43), for any z and z' in the interior of X, with $z < z'$, any $x,y \in X$ such that $x \geq y \geq z'$ and every $p \in [0,1]$, we have

$$u_z^{-1}\left(w_z(p)u_z(x) + (1-w_z(p))u_z(y)\right) = u_{z'}^{-1}\left(w_{z'}(p)u_{z'}(x) + (1-w_{z'}(p))u_{z'}(y)\right)$$

and so, according to Lemma 5, we get $w_z = w_{z'} =: w$ and

$$u_z(x) = \alpha u_{z'}(x) + \beta \quad \text{for} \quad x \geq z' \tag{44}$$

with some $\alpha \in (0,\infty)$ and $\beta \in \mathbb{R}$.

Let $(a_n : n \in \mathbb{N})$ be a decreasing sequence of elements of the interior of X such that $\lim_{n\to\infty} a_n = \inf X$. Moreover, let $a_0 \in X$ be such that $a_1 < a_0$. It follows from (43) that

$$u_{a_n}^{-1}(u_{a_n}(a_n) + w(p)(u_{a_n}(a_0) - u_{a_n}(a_n))) = F(a_0, a_n; p)$$
$$= u_{a_{n+1}}^{-1}(u_{a_{n+1}}(a_n) + w(p)(u_{a_{n+1}}(a_0) - u_{a_{n+1}}(a_n))) \quad \text{for} \quad n \in \mathbb{N},\ p \in [0,1].$$

Furthermore, as w is a continuous probability weighting function, for every $n \in \mathbb{N}$ there is a unique $p_n \in (0,1)$ such that $w(p_n) = \frac{u_{a_n}(a_1) - u_{a_n}(a_n)}{u_{a_n}(a_0) - u_{a_n}(a_n)}$. Therefore, we have

$$\frac{u_{a_{n+1}}(a_1) - u_{a_{n+1}}(a_n)}{u_{a_{n+1}}(a_0) - u_{a_{n+1}}(a_n)} = \frac{u_{a_n}(a_1) - u_{a_n}(a_n)}{u_{a_n}(a_0) - u_{a_n}(a_n)} \quad \text{for} \quad n \in \mathbb{N}$$

and so

$$\frac{u_{a_{n+1}}(a_n) - u_{a_{n+1}}(a_1)}{u_{a_{n+1}}(a_0) - u_{a_{n+1}}(a_1)} = \frac{u_{a_n}(a_n) - u_{a_n}(a_1)}{u_{a_n}(a_0) - u_{a_n}(a_1)} \quad \text{for} \quad n \in \mathbb{N}.$$

Thus, since for any $n \in \mathbb{N}$, u_{a_n} is a utility function, in view of (46) we get

$$\lim_{x\to a_n^-} \frac{u_{a_{n+1}}(x) - u_{a_{n+1}}(a_1)}{u_{a_{n+1}}(a_0) - u_{a_{n+1}}(a_1)} = \frac{u_{a_{n+1}}(a_n) - u_{a_{n+1}}(a_1)}{u_{a_{n+1}}(a_0) - u_{a_{n+1}}(a_1)} = \frac{u_{a_n}(a_n) - u_{a_n}(a_1)}{u_{a_n}(a_0) - u_{a_n}(a_1)}. \tag{45}$$

Define a function $u : X \setminus \{\inf X\} \to \mathbb{R}$ in the following way

$$u(x) = u_{a_1}(x) \quad \text{for} \quad x \geq a_1,$$
$$u(x) = \frac{u_{a_{n+1}}(x) - u_{a_{n+1}}(a_1)}{u_{a_{n+1}}(a_0) - u_{a_{n+1}}(a_1)} \quad \text{for} \quad x \in [a_{n+1}, a_n),\ n \in \mathbb{N}. \tag{46}$$

Then, in view of (45)–(46), for any $n \in \mathbb{N}$, we have $\lim_{x\to a_n^-} u(x) = u(a_n)$, i.e. u is continuous on $[a_{n+1}, a_n]$ and so, it is continuous. Furthermore, as u is strictly increasing on $[a_{n+1}, a_n]$ for $n \in \mathbb{N}$, it is strictly increasing. Therefore, u is a utility function.

We show that (1) holds. To this end fix $x, y \in X \setminus \{\inf X\}$, with $x \geq y$, and $p \in [0,1]$. Let $m = \min\{n \in \mathbb{N} : a_n \leq y\}$, $k = \min\{n \in \mathbb{N} : a_n \leq F(x,y;p)\}$, and $l = \min\{n \in \mathbb{N} : a_n \leq x\}$. Since $y \leq F(x,y;p) \leq x$, we get $a_m \leq a_k \leq a_l$. Then making use of (44) we obtain that there exist $\alpha, \gamma \in (0,\infty)$ and $\beta, \delta \in \mathbb{R}$ such that

$$u_{a_l}(x) = \alpha u_{a_m}(x) + \beta \quad \text{for} \quad x \geq a_l, \tag{47}$$
$$u_{a_k}(x) = \gamma u_{a_m}(x) + \delta \quad \text{for} \quad x \geq a_k. \tag{48}$$

Hence, successively applying (46), (48), (43), (47) and again (46), we obtain

$$\begin{aligned} u(F(x,y;p)) &= \frac{u_{a_k}(F(x,y;p)) - u_{a_k}(a_1)}{u_{a_k}(a_0) - u_{a_k}(a_1)} = \frac{u_{a_m}(F(x,y;p)) - u_{a_m}(a_1)}{u_{a_m}(a_0) - u_{a_m}(a_1)} \\ &= w(p)\frac{u_{a_m}(x) - u_{a_m}(a_1)}{u_{a_m}(a_0) - u_{a_m}(a_1)} + (1 - w(p))\frac{u_{a_m}(y) - u_{a_m}(a_1)}{u_{a_m}(a_0) - u_{a_m}(a_1)} \\ &= w(p)\frac{u_{a_l}(x) - u_{a_l}(a_1)}{u_{a_l}(a_0) - u_{a_l}(a_1)} + (1 - w(p))\frac{u_{a_m}(y) - u_{a_m}(a_1)}{u_{a_m}(a_0) - u_{a_m}(a_1)} \\ &= w(p)u(x) + (1 - w(p))u(y) \end{aligned}$$

□

The proofs of our main Theorems are divided into steps for better readability and clarity.

Proof of Theorem 1. Note that uniqueness follows directly from Lemma 3. In the existence part, we only prove the sufficiency of the axioms, because their necessity is obvious. We assume that F satisfies **(Ref)**, **(CM)** and **(Perm)**.

Step 1. We show that since y_0 is the endpoint of X we may restrict attention to the case $y_0 = \min X$. In fact, suppose that in this case the representation (3) holds. Note that, if $y_0 = \max X$, then $\tilde{X} := \{2y_0 - x : \ x \in X\}$ is a real interval with $y_0 = \min \tilde{X}$ and a function $\tilde{F} : \Delta_{y_0}(\tilde{X}) \to \tilde{X}$, given by

$$\tilde{F}(x,p) = 2y_0 - F(2y_0 - x, p) \quad \text{for} \quad (x,p) \in \Delta_{y_0}(\tilde{X}),$$

satisfies **(Ref)**, **(CM)** and **(Perm)**. In fact, **(Ref)** and **(Perm)** are easy to verify and, because F is continuous and strictly increasing in payoff, $\tilde{F}$ has the same properties. Moreover, we have $2y_0 - x < y_0 < x$ for $x \in \tilde{X} \setminus \{y_0\}$, and so F is strictly decreasing in the probability of $2y_0 - x$. Hence, $\tilde{F}$ is strictly increasing in the probability of x and thus fulfills **(CM)**. We conclude that there exist a continuous probability weighting function w and a utility function $\tilde{u} : \tilde{X} \to \mathbb{R}$ satisfying $\tilde{u}(y_0) = 0$ such that

$$\tilde{F}(x,p) = \tilde{u}^{-1}(w(p)\tilde{u}(x)) \quad \text{for} \quad (x,p) \in \Delta_{y_0}(\tilde{X}).$$

Then $u : X \to \mathbb{R}$, defined by $u(x) = -\tilde{u}(2y_0 - x)$ for $x \in X$, is a utility function satisfying $u(y_0) = 0$ and for any $(x,p) \in \Delta_{y_0}(X)$, we get

$$F(x,p) = 2y_0 - \tilde{F}(2y_0 - x, p) = u^{-1}(-w(p)\tilde{u}(2y_0 - x)) = u^{-1}(w(p)u(x)),$$

that is the representation (3) holds. From now on we assume that $y_0 = \min X$.

Step 2. We show that for any $p \in (0,1)$, $F(x,p)$ is continuous and strictly increasing in x. Let $p \in (0,1)$ and $x \in X$. First, assume that x is an interior point of X. Thus, taking $y \in X$ with $x < y$, we get $x = F(y,q)$ for some $q \in (0,1)$. Furthermore, for any sequence $(x_n : n \in \mathbb{N})$ sequence of elements of the interval (y_0, y) converging to x there exists a corresponding sequence $(q_n : n \in \mathbb{N})$ of elements of $(0,1)$ such that $x_n = F(y, q_n)$ for $n \in \mathbb{N}$. Thus, in view of **(CM)**, we get

$$F(y,q) = x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} F(y,q_n) = F(y, \lim_{n\to\infty} q_n)$$

and so $\lim_{n\to\infty} q_n = q$. Hence, making use of **(Perm)** and **(CM)**, we obtain

$$\lim_{n\to\infty} F(x_n,p) = \lim_{n\to\infty} F(F(y,q_n)),p) = \lim_{n\to\infty} F(F(y,p)),q_n)$$

$$= F(F(y,p),q) = F(F(y,q),p) = F(x,p).$$

If $x = y_0$ or $x = \max X$ then the same reasoning shows a right (left, respectively) continuity at x. This proves the continuity of $F(x,p)$ in x. We now prove the monotonicity. To this end fix $x_1, x_2 \in X$ with $y_0 \le x_1 < x_2$. Then, in view of **(CM)**, we get $x_1 = F(x_2, q)$ for some $q \in (0,1)$. Hence, applying **(Ref)**, **(CM)** and **(Perm)**, for every $p \in (0,1)$, we obtain

$$F(x_1,p) = F(F(x_2,q),p) = F(F(x_2,p),q) < F(F(x_2,p),1) = F(F(x_2,p)) = F(x_2,p).$$

Thus $F(x,p)$ is strictly monotone in x for any $p \in (0,1)$.

Step 3. We now derive the representation. If $\sup X \in X$ then put $b = \sup X$. Otherwise, let b be an arbitrary element of the interior of X. It follows from **(Ref)** that $F(b,0) = F(y_0) = y_0 < b = F(b) = F(b,1)$. Thus, in view of **(CM)**, for every $x \in [y_0, b]$ there exists a unique $p_x \in [0,1]$ such that $F(b, p_x) = x$. Applying the idea in Hosszú (1962) (cf. Aczél, 1966, pp. 270–271), define in $(y_0, b]$ a binary operation $\star$ in the following way

$$x \star y = F(y, p_x) \quad \text{for} \quad x, y \in (y_0, b]. \tag{49}$$

First, we show that $\star$ is commutative, associative, cancellative and continuous. To see that $\star$ is commutative, for any $x, y \in (y_0, b]$ apply **(Perm)**, to get

$$x \star y = F(y, p_x) = F(F(b, p_y), p_x) = F(F(b, p_x), p_y) = F(x, p_y) = y \star x.$$

Using commutativity of $\star$ and **(Perm)**, for every $x, y, z \in (y_0, b]$, we obtain

$$x \star (y \star z) = x \star (z \star y) = F(z \star y, p_x) = F(F(y, p_z), p_x)$$

$$= F(F(y, p_x), p_z) = F(x \star y, p_z) = z \star (x \star y) = (x \star y) \star z,$$

which proves that $\star$ is associative. To show the cancellativity of $\star$, suppose that $x \star z = y \star z$ for some $x, y, z \in (y_0, b]$. Then $F(z, p_x) = F(z, p_y)$ and so, taking **(Perm)** into account, we get

$$F(x, p_z) = F(F(b, p_x), p_z) = F(F(b, p_z), p_x) = F(z, p_x)$$

$$= F(z, p_y) = F(F(b, p_z), p_y) = F(F(b, p_y), p_z) = F(y, p_z).$$

Hence, in view of Step 2, we obtain $x = y$. In this way we have proved that $\star$ is right-cancellative. By commutativity, $\star$ is also left-cancellative and hence cancellative. Continuity of $\star$ follows from **(CM)** and Step 2. Thus we have proved that $\star$ possesses the required properties. Hence, applying Craigen and Páles (1989), we conclude that there exist an unbounded real interval I and a continuous bijection $f : (y_0, b] \to I$ such that

$$x \star y = f^{-1}(f(x) + f(y)) \quad \text{for} \quad x, y \in (y_0, b]. \tag{50}$$

Since replacing f by $-f$ does not alter (50), we may assume that f is strictly increasing. Note that $p_{F(b,p)} = p$ for $p \in [0,1]$, and so it follows from (49) that

$$F(x, p) = F(b, p) \star x \quad \text{for} \quad x \in (y_0, b],\ p \in (0, 1].$$

Thus applying (50) on the right hand side yields

$$F(x, p) = f^{-1}(f(F(b, p)) + f(x)) \quad \text{for} \quad x \in (y_0, b],\ p \in (0, 1], \tag{51}$$

Setting $p = 1$ in (51), in view of **(Ref)**, we get

$$f(b) = 0. \tag{52}$$

Hence, as $f:(y_0,b]\to I$ is an increasing bijection and I is unbounded, we conclude that $I=(-\infty,0]$ and $\lim_{x\to y_0^+} f(x)=-\infty$. Therefore, $u:[y_0,b]\to\mathbb{R}$ given by

$$u(x)=\begin{cases} e^{f(x)} & \text{for } x\in(y_0,b],\\ 0 & \text{for } x=y_0,\end{cases} \tag{53}$$

is a strictly increasing continuous function with $u(y_0)=0$. Moreover, in view of **(CM)** and (52), $w:[0,1]\to[0,1]$ defined by

$$w(p)=\begin{cases} e^{f(F(b,p))} & \text{for } p\in(0,1],\\ 0 & \text{for } p=0,\end{cases} \tag{54}$$

is a continuous probability weighting function. From (51), (53) and (54) we derive that

$$F(x,p)=u^{-1}(w(p)u(x)) \quad \text{for} \quad x\in[y_0,b],\ p\in[0,1].$$

If $\sup X=X$, this gives a required representation. If $\sup X\notin X$, then as b is an arbitrary element of $[y_0,a)$, applying Lemma 4, we get the assertion. □

In the sequel, we will use the following notation: $X_{\le y_0}:=X\cap(-\infty,y_0]$ and $X_{\ge y_0}:=X\cap[y_0,\infty)$. Similarly, we set $X_{<y_0}:=X\cap(-\infty,y_0)$ and $X_{>y_0}:=X\cap(y_0,\infty)$.

Proof of Theorem 2. If $F:\Delta_{y_0}(X)\to X$ satisfies **(Ref)**, **(CM)** and **(Perm)**, then applying Theorem 1 twice (first with X replaced by $X_{\le y_0}$, and then with X replaced by $X_{\ge y_0}$), we obtain the existence of continuous probability weighting functions w_-,w_+ and utility functions $u_-:X_{\le y_0}\to\mathbb{R}$ and $u_+:X_{\ge y_0}\to\mathbb{R}$ such that $u_-(y_0)=u_+(y_0)=0$ and

$$F(x,p)=\begin{cases} u_-^{-1}(w_-(p)u_-(x)) & \text{for } x<y_0,\ p\in[0,1],\\ u_+^{-1}(w_+(p)u_+(x)) & \text{for } x\ge y_0,\ p\in[0,1].\end{cases}$$

This yields the representation (6) with $u:X\to\mathbb{R}$ given by

$$u(x)=\begin{cases} u_-(x) & \text{for } x<y_0,\\ u_+(x) & \text{for } x\ge y_0.\end{cases}$$

Note also that, as u_- and u_+ are utility functions with $u_+(y_0)=u_-(y_0)=0$, u is a utility function with $u(y_0)=0$. This completes the existence part of the proof.

We now prove uniqueness. Assume that (6) is satisfied with w_-,w_+ replaced by another pair of probability weighting functions $\tilde{w}_-,\tilde{w}_+$, and u replaced by another utility function $\tilde{u}:X\to\mathbb{R}$ satisfying $\tilde{u}(y_0)=0$. Let u_- and u_+ be the restrictions of u to $X_{\le y_0}$ and $X_{\ge y_0}$, respectively. Similarly, let $\tilde{u}_-$ and $\tilde{u}_+$ be the corresponding restrictions of $\tilde{u}$. Then we get

$$\begin{aligned} \tilde{u}_-^{-1}(\tilde{w}_-(p)\tilde{u}_-(x))&=u_-^{-1}(w_-(p)u_-(x)) \quad \text{for} \quad x\in X_{\le y_0},\ p\in[0,1],\\ \tilde{u}_+^{-1}(\tilde{w}_+(p)\tilde{u}_+(x))&=u_+^{-1}(w_+(p)u_+(x)) \quad \text{for} \quad x\in X_{\le y_0},\ p\in[0,1].\end{aligned}$$

Therefore, applying Lemma 3, we obtain that there exist $\alpha,\beta,r_-,r_+\in(0,\infty)$ such that (7) and (8) hold, $\tilde{u}_-(x)=-\alpha(-u_-(x))^{r_-}$ for $x\in X_{\le y_0}$ and $\tilde{u}_+(x)=\beta u_+(x)^{r_+}$ for $x\in X_{\ge y_0}$. Hence, (9) holds which concludes the uniqueness part of the proof of Theorem 2. □

Proof of Theorem 3. The uniqueness part follows from the uniqueness parts of Theorems 1 and 2 in the respective two cases, with the additional restriction that the weighting functions acting in these theorems are the identity on $[0,1]$. We now prove the existence part. Necessity of the axioms is obvious. In order to prove their sufficiency assume that **(Ref)**, **(CM)** and **(Red)** hold. First assume that y_0 is the endpoint of X. Then, as **(Red)** implies **(Perm)**, applying Theorem 1 there exist a continuous probability weighting function w and a utility function $u : X \to \mathbb{R}$ satisfying $u(y_0) = 0$ such that F is of the form (3). Plugging it into **(Red)** we obtain

$$w(pq) = w(p)w(q) \quad \text{for} \quad p, q \in [0,1].$$

Hence, since w is continuous, by the standard result (see for example Kuczma, 2008, Theorem 13.1.6) there exists $\alpha > 0$ such that $w(p) = p^\alpha$ for $p \in [0,1]$. Therefore, defining $\tilde{u} : X \to \mathbb{R}$ by $\tilde{u}(x) = u(x)^{1/\alpha}$ for $x \in X$, and taking (3) into account, we conclude that $F(x,p) = \tilde{u}^{-1}(p\tilde{u}(x))$ for $x \in X$ and $p \in [0,1]$. This yields the required representation in the case when y_0 is the endpoint of X. If y_0 is the interior point of X then, as **(Red)** implies **(Perm)**, according to Theorem 2, F is of the form (6) with some continuous probability weighting functions w_-, w_+, and a utility function $u : X \to \mathbb{R}$ satisfying $u(y_0) = 0$. Similarly as before we thus obtain that

$$w_i(pq) = w_i(p)w_i(q) \quad \text{for} \quad p, q \in [0,1],\ i \in \{+,-\},$$

which yields that $w_i(p) = p^{\alpha_i}$, $p \in [0,1]$ for some $\alpha_i > 0$. Hence the required representation holds with $\tilde{u} : X \to \mathbb{R}$ given by

$$\tilde{u}(x) = \begin{cases} -(-u(x))^{1/\alpha_-} & \text{for} \quad x \le y_0, \\ u(x)^{1/\alpha_+} & \text{for} \quad x \ge y_0. \end{cases} \tag{55}$$

□

Proof of Theorem 4. The 'if' part of the uniqueness is straightforward. The 'only if' part follows directly from Lemma 5. That the axioms are necessary for the representation is clear.

We now prove their sufficiency. Assume that **(Ref)**, **(CM)** and **(Dist)** hold. Let y_1 be an arbitrarily fixed element of the interior of X. The remaining part of the proof is divided into three steps. In Step 1, we show that the axioms imply a special case of the functional equation analyzed by Gilányi et al. (2005). In Step 2, we use their solution to establish a required representation for prospects with payoffs in $X_{\ge y_1}$. Finally, the representation for arbitrary prospects in $\Delta(X)$ is derived in Step 3.

Step 1. Let y_0 be an arbitrary element of X such that $y_0 < y_1$. For $i \in \{0,1\}$ define the function $F_i : \Delta_{y_i}(X_{\ge y_i}) \to X_{\ge y_i}$ by

$$F_i(x;p) = F(x, y_i; p) \quad \text{for} \quad (x;p) \in \Delta_{y_i}(X_{\ge y_i}). \tag{56}$$

Setting in **(Dist)** $y = z = y_i$ for $i \in \{0,1\}$, in view of **(Ref)**, we get

$$F_i(F_i(x,p);q) = F_i(F_i(x,q);p) \quad \text{for} \quad x \in X_{\ge y_i},\ p \in [0,1],\ i \in \{0,1\}. \tag{57}$$

Thus, making use of **(CM)** and applying Theorem 1, we obtain that for $i \in \{0,1\}$ there exist a continuous probability weighting function w_i and a utility function $v_i : X_{\ge y_i} \to \mathbb{R}$ such that $v_i(y_i) = 0$ and

$$F_i(x;p) = v_i^{-1}(w_i(p)v_i(x)) \quad \text{for} \quad x \in X_{\ge y_i},\ p \in [0,1],\ i \in \{0,1\}. \tag{58}$$

Hence $v_0(y_1) > v_0(y_0) = 0$ and so, normalizing v_0, we conclude that for $\bar{v}_0 := \frac{v_0}{v_0(y_1)}$, we have

$$\bar{v}_0(y_0) = 0, \quad \bar{v}_0(y_1) = 1 \tag{59}$$

and

$$F_0(x;p) = \bar{v}_0^{-1}(w_0(p)\bar{v}_0(x)) \quad \text{for} \quad x \in X_{\geq y_0},\ p \in [0,1]. \tag{60}$$

Furthermore, setting in **(Dist)** $z = y_0$, in view of (56) and (60), we get

$$\bar{v}_0^{-1}(w_0(q)\bar{v}_0(F(x,y;p))) = F(\bar{v}_0^{-1}(w_0(q)\bar{v}_0(x)), \bar{v}_0^{-1}(w_0(q)\bar{v}_0(y)); p)$$

for $x, y \in X_{\geq y_0}$ with $x \geq y$ and $p, q \in [0,1]$. Replacing in this equality x and y by $\bar{v}_0^{-1}(x)$ and $\bar{v}_0^{-1}(y)$, respectively, we conclude that

$$W_p(w_0(q)x, w_0(q)y) = w_0(q)W_p(x,y) \ \text{ for } \ x, y \in \bar{v}_0(X_{\geq y_0}),\ x \geq y,\ p, q \in [0,1], \tag{61}$$

where for any $p \in [0,1]$ a function $W_p : \bar{v}_0(X_{\geq y_0})^2 \to \mathbb{R}$ is given by

$$W_p(x,y) = \bar{v}_0(F(\bar{v}_0^{-1}(x), \bar{v}_0^{-1}(y); p)) \quad \text{for} \quad x, y \in \bar{v}_0(X_{\geq y_0}). \tag{62}$$

Since w_0 is a continuous probability weighting function, we have $\{w_0(p) : p \in [0,1]\} = [0,1]$ and so it follows from (61) that

$$W_p(\lambda x, \lambda y) = \lambda W_p(x,y) \quad \text{for} \quad x, y \in \bar{v}_0(X_{\geq y_0}),\ x \geq y,\ \lambda \in [0,1],\ p \in [0,1]. \tag{63}$$

Moreover, in view of (59), we have $\frac{1}{\bar{v}_0(y)} \in (0,1]$ for $y \in X_{\geq y_1}$. Hence, applying (63), for every $x, y \in X_{\geq y_1}$ with $x \geq y$ and $p \in [0,1]$, we obtain

$$W_p(\bar{v}_0(x), \bar{v}_0(y)) = \bar{v}_0(y)\frac{1}{\bar{v}_0(y)}W_p(\bar{v}_0(x), \bar{v}_0(y)) = \bar{v}_0(y)W_p\left(\frac{\bar{v}_0(x)}{\bar{v}_0(y)}, 1\right).$$

Thus, taking (62) into account, for every $x, y \in X_{\geq y_1}$, with $x \geq y$, and $p \in [0,1]$, we get

$$F(x,y;p) = \bar{v}_0^{-1}(W_p(\bar{v}_0(x), \bar{v}_0(y))) = \bar{v}_0^{-1}\left(\bar{v}_0(y)W_p\left(\frac{\bar{v}_0(x)}{\bar{v}_0(y)}, 1\right)\right).$$

Hence

$$F(x,y;p) = \bar{v}_0^{-1}\left(\bar{v}_0(y)\Phi_p\left(\frac{\bar{v}_0(x)}{\bar{v}_0(y)}\right)\right) \quad \text{for} \quad x, y \in X_{\geq y_1},\ x \geq y,\ p \in [0,1], \tag{64}$$

where, for every $p \in [0,1]$, a function $\Phi_p : I \to \mathbb{R}$ is given by

$$\Phi_p(s) = W_p(s, 1) \quad \text{for} \quad s \in I, \tag{65}$$

with $I := \bar{v}_0(X_{\geq y_1})$. Note that, as $\bar{v}_0$ is a utility function and $\bar{v}_0(y_1) = 1$, I is a real interval containing its left endpoint 1. Furthermore, plugging (64) into **(Dist)** with $z = y_1$, in view of (56) and (59), for every $x, y \in X$, with $x \geq y \geq y_1$ and $p, q \in [0,1]$, we get

$$\Phi_q\left(\bar{v}_0(y)\Phi_p\left(\frac{\bar{v}_0(x)}{\bar{v}_0(y)}\right)\right) = \Phi_q\left(\bar{v}_0(y)\right)\Phi_p\left(\frac{\Phi_q\left(\bar{v}_0(x)\right)}{\Phi_q\left(\bar{v}_0(y)\right)}\right).$$

Hence, we have

$$\frac{\Phi_q\left(t\Phi_p\left(\frac{s}{t}\right)\right)}{\Phi_q(t)} = \Phi_p\left(\frac{\Phi_q(s)}{\Phi_q(t)}\right) \quad \text{for} \quad s,t\in I,\ s\geq t,\ p,q\in[0,1]. \tag{66}$$

Since $\bar{v}_0(y_1)=1$, applying (65), (62), (56) and (57) successively, we obtain

$$\begin{aligned}\Phi_p(s) &= W_p(s,1) = \bar{v}_0(F(\bar{v}_0^{-1}(x),\bar{v}_0^{-1}(1);p)) = \bar{v}_0(F(\bar{v}_0^{-1}(s),y_1;p))\\ &= \bar{v}_0(F_1(\bar{v}_0^{-1}(s);p)) = \bar{v}_0(v_1^{-1}(w_1(p)v_1(\bar{v}_0^{-1}(s)))) \quad \text{for} \quad s\in I,\ p\in[0,1].\end{aligned}$$

Therefore, setting $\Phi := v_1\circ\bar{v}_0^{-1}$, we get

$$\Phi_p(s) = \Phi^{-1}(w_1(p)\Phi(s)) \quad \text{for} \quad s\in I,\ p\in[0,1]. \tag{67}$$

Note that, as I is an interval containing its left endpoint 1, the interior of I is of the form $(1,d)$ with some $1<d\leq\infty$. We show that for any $q\in[0,1]$ and $t\in(1,d)$ a function $f_{(q,t)}:\left(1,\frac{d}{t}\right)\to\mathbb{R}$ defined in the following way

$$f_{(q,t)}(x) = \frac{1}{\Phi(x)}\Phi\left(\frac{\Phi_q(tx)}{\Phi_q(t)}\right) \quad \text{for} \quad x\in\left(1,\frac{d}{t}\right), \tag{68}$$

is constant. Fix $q\in[0,1]$, $t\in(1,d)$ and $x_1,x_2\in\left(1,\frac{d}{t}\right)$ with $x_1<x_2$. Let $s\in(tx_2,d)$. Then $x_i<\frac{s}{t}$ for $i\in\{1,2\}$ and so, as Φ is strictly increasing, with $\Phi(1)=(v_1\circ\bar{v}_0^{-1})(1)=v_1(y_1)=0$, we have $\frac{\Phi(x_i)}{\Phi\left(\frac{s}{t}\right)}\in(0,1)$ for $i\in\{1,2\}$. Thus, since w_1, being a continuous probability weighting function, is onto $[0,1]$, for $i\in\{1,2\}$ there exists $p_i\in(0,1)$ such that $w_1(p_i)=\frac{\Phi(x_i)}{\Phi\left(\frac{s}{t}\right)}$. Hence, in view of (67), for $i\in\{1,2\}$, we get

$$\Phi_{p_s}\left(\frac{s}{t}\right) = \Phi^{-1}\left(w_1(p_1)\Phi\left(\frac{s}{t}\right)\right) = x_i$$

and

$$(\Phi\circ\Phi_{p_i})\left(\frac{\Phi_q(s)}{\Phi_q(t)}\right) = w_1(p_i)\Phi\left(\frac{\Phi_q(s)}{\Phi_q(t)}\right) = \frac{\Phi\left(\frac{\Phi_q(s)}{\Phi_q(t)}\right)}{\Phi\left(\frac{s}{t}\right)}\Phi(x_i).$$

Applying Φ on both sides of (66) with $p=p_i$, and making use of the above two equalities we obtain

$$\frac{1}{\Phi(x_i)}\Phi\left(\frac{\Phi_q(tx_i)}{\Phi_q(t)}\right) = \frac{\Phi\left(\frac{\Phi_q(s)}{\Phi_q(t)}\right)}{\Phi\left(\frac{s}{t}\right)} \quad \text{for} \quad i\in\{1,2\}.$$

Thus, taking (68) into account, we conclude that $f_{(q,t)}(x_1)=f_{(q,t)}(x_2)$, and hence $f_{(q,t)}$ is constant, say $f_{(q,t)}(x)=c(q,t)$ for $x\in\left(1,\frac{d}{t}\right)$, with some $c(q,t)\in\mathbb{R}$. So, in view of (68), for any $q\in[0,1]$, $t\in(1,d)$ and $x\in\left(1,\frac{d}{t}\right)$, we have

$$c(q,t) = \frac{1}{\Phi(x)}\Phi\left(\frac{\Phi_q(tx)}{\Phi_q(t)}\right) > \frac{\Phi(1)}{\Phi(x)} = 0$$

and

$$\Phi\left(\frac{\Phi_q(tx)}{\Phi_q(t)}\right) = c(q,t)\Phi(x).$$

Therefore, for any $q \in [0,1]$, we obtain

$$\Psi(H_q(\ln t) - H_q(\ln t + \ln x)) = G_q(\ln t) + \Psi(\ln x) \quad \text{for} \quad t, x \in (1,d),\ tx \in (1,d),$$

where $G_q, H_q, \Psi : (0, \ln d) \to \mathbb{R}$ are given by

$$H_q(z) = -\ln \Phi_q(e^z) \quad \text{for} \quad z \in (0, \ln d), \tag{69}$$

$$G_q(z) = \ln c(q, e^z) \quad \text{for} \quad z \in (0, \ln d), \tag{70}$$

$$\Psi(y) = \ln \Phi(e^y) \quad \text{for} \quad y \in (0, \ln d), \tag{71}$$

with a convention $\ln \infty = \infty$. Thus, for every $q \in [0,1]$, we have

$$H_q(z) - H_q(z+y) = \Psi^{-1}\left(G_q(z) + \Psi(y)\right) \quad \text{for} \quad z, y \in (0, \ln d),\ z + y \in (0, \ln d). \tag{72}$$

Note that, since Φ and Φ_q for $q \in (0,1)$ are continuous, it follows from (69) and (71) that so are Ψ and H_q for $q \in (0,1)$. Thus, in view of (72), G_q is continuous for any $q \in (0,1)$.

Step 2. Equation (72) is a particular case of the functional equation analyzed by Gilányi et al. (2005). Thus, according to their Theorem 2, for every $q \in [0,1]$, G_q is either constant or it is strictly monotone. We will first consider the case where G_q is strictly monotone for some $q \in [0,1]$, and then the case where G_q is constant for every $q \in [0,1]$.

Assume that G_q is strictly monotone for some $q \in [0,1]$. Then, as Ψ is strictly increasing, according to (Gilányi et al., 2005, Theorem 2), either there exist $\alpha \in \mathbb{R} \setminus \{0\}$, $\beta \in (0, \infty)$ and $\gamma \in \mathbb{R}$ such that

$$\Psi(x) = \beta \ln \left|1 - e^{-\alpha x}\right| + \gamma \quad \text{for} \quad x \in (0, \ln d), \tag{73}$$

or there exist $\beta \in (0, \infty)$ and $\gamma \in \mathbb{R}$ such that

$$\Psi(x) = \beta \ln x + \gamma \quad \text{for} \quad x \in (0, \ln d). \tag{74}$$

If (73) holds, then as ϕ is continuous with $\Phi(1) = 0$, in view of (71), we get

$$\Phi(x) = e^{\gamma} \left|1 - x^{-\alpha}\right|^{\beta} \quad \text{for} \quad x \in I.$$

Therefore, considering separately the case of α negative and then α positive and in each of them applying first (67) and then (64), for every $x, y \in X_{\geq y_1}$ with $x \geq y$ and $p \in [0,1]$, we get

$$F(x,y;p) = \bar{v}_0^{-1}\left(\left(w_1(p)^{\frac{1}{\beta}} \bar{v}_0(x)^{-\alpha} + (1 - w_1(p)^{\frac{1}{\beta}})\bar{v}_0(y)^{-\alpha}\right)^{-\frac{1}{\alpha}}\right). \tag{75}$$

Define $w := w_1^{\frac{1}{\beta}}$ and $u_0 : X_{\geq y_1} \to \mathbb{R}$ as

$$u_0(x) = |\bar{v}_0(x)^{-\alpha} - 1| \quad \text{for} \quad x \in X_{\geq y_1}.$$

Then w is a continuous probability weighting function and u_0 is a utility function with $u_0(y_1) = 0$. Furthermore, in view of (75), we have

$$F(x,y;p) = u_0^{-1}\left(w(p)u_0(x) + (1 - w(p))u_0(y)\right) \quad \text{for} \quad x \geq y \geq y_1,\ p \in [0,1]. \tag{76}$$

If (74) holds, then the functional equation (72) becomes

$$H_q(z) - H_q(z+y) = e^{\frac{1}{\beta}G_q(z)}y \quad \text{for} \quad z, y \in (0, \ln d),\ z + y < \ln d. \tag{77}$$

Since H_q is continuous and $H_q(0) = 0$, it follows from (77) that

$$\frac{H_q(y)}{y} = -\lim_{z \to 0^+} e^{\frac{1}{\beta}G_q(z)} =: a \quad \text{for } y \in (0, \ln d).$$

Therefore $H_q(y) = ay$ for $y \in (0, \ln d)$ and so from (77) we derive that $e^{\frac{1}{\beta}G_q(z)} = -a$ for $z \in (0, \ln d)$, which contradicts the strict monotonicity of G_q.

We now consider the case where G_q is constant for every $q \in [0, 1]$. Then, as for every $q \in [0, 1]$, H_q is a strictly decreasing continuous function with $H_q(0) = 0$, applying again (Gilányi et al., 2005, Theorem 2), we conclude that for every $q \in [0, 1]$ there exists a $w(q) \in (0, \infty)$ such that $H_q(x) = -w(q)x$ for $x \in (0, \ln d)$. Thus, in view of (69) and the fact that Φ_q is continuous for every $q \in [0, 1]$, we get

$$\Phi_q(x) = x^{w(q)} \quad \text{for} \quad x \in I,\ q \in [0, 1]. \tag{78}$$

Hence, taking (64) into account, we obtain

$$F(x, y; p) = \bar{v}_0^{-1}\left(\bar{v}_0(x)^{w(p)}\bar{v}_0(y)^{1-w(p)}\right) \quad \text{for} \quad x \geq y \geq y_1,\ p \in [0, 1].$$

This yields (76) with $u_0 : X_{\geq y_1} \to \mathbb{R}$ given by $u_0(x) = \ln \bar{v}_0(x)$ for $x \in X_{\geq y_1}$. Note that u_0 is a utility function, with $u_0(y_1) = 0$. Moreover, it follows from **(CM)** and (76) that w is a continuous probability weighting function.

This concludes the analysis of all possible cases. In one of them a contradiction was derived, while in the other two we obtained the required representation (76).

Step 3. Since y_1 was an arbitrary element in the interior of X, applying Lemma 6, we conclude that if $\inf X \notin X$ the representation (1) holds with some utility function u and a continuous probability weighting function w.

Assume that $\ell := \inf X \in X$. Then, in view of **(CM)**, for every $x \in X \setminus \{\ell\}$ and $p \in (0, 1)$, we have $F(x, \ell; p) \in X \setminus \{\ell\}$ and so we get

$$c := \lim_{y \to \ell^+} u(y) = \lim_{y \to \ell^+} \frac{u(F(x, y; p)) - w(p)u(x)}{1 - w(p)} = \frac{u(F(x, \ell; p)) - w(p)u(x)}{1 - w(p)}.$$

Therefore, extending u to X by putting $u(\ell) = c$, we conclude that u is a utility function on X. Furthermore, for every $x \in X \setminus \{\ell\}$ and $p \in (0, 1)$, we have

$$F(x, \ell; p) = u^{-1}(w(p)u(x) + (1 - w(p))c) = u^{-1}(w(p)u(x) + (1 - w(p))u(\ell)).$$

Obviously, in view of **(Ref)**, the last equality holds also for $x = \ell$ or $p \in \{0, 1\}$. Thus, a proof of the representation (1) is completed. □

Proof of Theorem 5. The uniqueness part is standard. In fact, it follows from Theorem 4. That the axioms are necessary for the existence of the representation is obvious. We now prove their sufficiency.

Assume that $F : \Delta(X) \to X$ satisfies **(Ref)**, **(CM)** and **(Red2)**. Note that **(Red2)** is a system of **(Red)** indexed by $y \in X$. Therefore, according to Theorem 3, for any $y \in X$, there exists a utility function $u_y : X \to \mathbb{R}$ such that $u_y(y) = 0$ and

$$F(x, y; p) = u_y^{-1}(pu_y(x)) \quad \text{for} \quad x \in X,\ p \in [0, 1]. \tag{79}$$

If $\inf X \in X$ then put $z = \inf X$. Otherwise, let z be an arbitrary element of the interior of X. Then, setting $u := u_z$, in view of (2) and (79), we get

$$u_y^{-1}(pu_y(z)) = F(z, y; p) = F(y, z; 1 - p) = u^{-1}((1 - p)u(y)) \quad \text{for} \quad y \in X,\ p \in [0, 1]. \tag{80}$$

Since $1 - \frac{u(x)}{u(y)} \in (0, 1]$ for $x, y \in X$ with $y > x \geq z$, applying (79) in the first equality and (80) in the third and fifth equalities, we obtain

$$\begin{aligned} F(x, y; p) &= u_y^{-1}(pu_y(x)) \\ &= u_y^{-1}\left(pu_y\left(u^{-1}\left(\left(1 - \left(1 - \tfrac{u(x)}{u(y)}\right)\right)u(y)\right)\right)\right) \\ &= u_y^{-1}\left(pu_y\left(u_y^{-1}\left(\left(1 - \tfrac{u(x)}{u(y)}\right)u_y(z)\right)\right)\right) \\ &= u_y^{-1}\left(p\left(1 - \tfrac{u(x)}{u(y)}\right)u_y(z)\right) \\ &= u^{-1}\left(\left(1 - p\left(1 - \tfrac{u(x)}{u(y)}\right)\right)u(y)\right) \\ &= u^{-1}(pu(x) + (1 - p)u(y)). \end{aligned}$$

Thus, in view of **(Ref)**, we get

$$F(x, y; p) = u^{-1}(pu(x) + (1 - p)u(y)) \quad \text{for} \quad x, y \in X_{\geq z},$$

which concludes the proof in the case $\inf X \in X$. If $\inf X \notin X$, then the assertion follows from Lemma 6. □

Proof of Corollary 1. Necessity of the axioms is obvious. We now prove their sufficiency. Assume that **(Ref)**, **(CM)**, and **($\overline{\textbf{Dist}}$)** hold. Since **($\overline{\textbf{Dist}}$)** implies **(Dist)**, by Theorem 4 we obtain the existence of a utility function u and a continuous probability weighting function w such that (1) holds. In view of (13), in order to get the required representation, it is enough to show that w satisfies (14), i.e. it is self-conjugate. Fix $x, y, z \in X$ with $x > z > y$ and $q \in [0, 1]$. According to **(CM)** and **(Ref)** there exists $p \in (0, 1)$ such that $F(x, y; p) > z$. Then applying (1), we get

$$F(F(x, y; p), z; q) = u^{-1}(w(q)w(p)u(x) + w(q)(1 - w(p))u(y) + (1 - w(q))u(z)).$$

Furthermore, using **(CM)** and **(Ref)** again, we obtain

$$y \leq F(y, z; q) \leq z \leq F(x, z; q) \leq x.$$

Thus, applying (1) and (13), yields

$$F(F(x, z; q), F(y, z; q); p)$$

$= u^{-1}(w(p)w(q)u(x)+(1-w(1-q))(1-w(p))u(y)+(w(p)(1-w(q))+(1-w(p)w(1-q))u(z))$.

Therefore, in view of **($\overline{\textbf{Dist}}$)** we get

$$(1 - w(p))(w(q) + w(1 - q) - 1)(u(z) - u(y)) = 0.$$

Since the first and the third part of the above product are strictly positive, the middle part must be zero. Since q is arbitrary, this proves that w is self-conjugate, which ends the proof. □

Proof of Theorem 6. That *(i)* implies *(iv)* is straightforward to verify. That *(iv)* implies *(ii)* and also *(iii)* is obvious. We now prove that *(ii)* implies *(i)*. Assume that (19) holds for every $p \in (0, 1)$ and $q = p$. Then, taking (2) into account, we get that (19) holds with p replaced by $1-p$ and q replaced by $q = 1-p$. Thus, applying (Aczél, 1947, Theorem of §2), we obtain the existence of utility functions u, v and weights $w(p), w(1-p) \in (0, 1)$ such that

$$F(x, y; p) = u^{-1}(w(p)u(x) + (1 - w(p))u(y)) \quad \text{for} \quad x, y \in X,\ p \in (0, 1) \tag{81}$$
$$F(x, y; 1 - p) = v^{-1}(w(1 - p)v(x) + (1 - w(1 - p))v(y)) \quad \text{for} \quad x, y \in X,\ p \in (0, 1).$$

Hence, in view of (2), for every $x, y \in X$ and $p \in (0, 1)$, we have

$$u^{-1}(w(p)u(x) + (1 - w(p))u(y) = v^{-1}((1 - w(1 - p))v(x) + w(1 - p)v(y)).$$

Therefore, applying Lemma 5, yields $w(p) = 1-w(1-p)$ for $p \in (0, 1)$. So, putting $w(0) = 0$ and $w(1) = 1$, and using the strict monotonicity of F in probability, we conclude that w is a self-conjugate probability weighting function. Moreover, making use of **(Ref)**, from (81) we deduce that *(i)* holds. It is left to show that *(iii)* implies *(i)*. Assume that (19) holds for every $p \in (0, 1)$ and some $q \in (0, 1)$. Then taking $p = q$ and applying (Aczél, 1947, Theorem of §2), we obtain that there exist a utility function $u : X \to \mathbb{R}$ and a $\beta \in (0, 1)$ such that

$$F(x, y; q) = u^{-1}(\beta u(x) + (1 - \beta)u(y)) \quad \text{for} \quad x, y \in X. \tag{82}$$

For every $p \in [0, 1]$ define a function $W_p : u(X)^2 \to u(X)$ by

$$W_p(s, t) = u(F(u^{-1}(s), u^{-1}(t); p)) \quad \text{for} \quad s, t \in u(X). \tag{83}$$

Then, in view of (19) and (82), for every $p \in (0, 1)$ and $(x_1, x_2, x_3, x_4) \in X^4$, we get

$$\begin{aligned}
&W_p\left(\beta(u(x_1), u(x_2)) + (1 - \beta)(u(x_3), u(x_4))\right)\\
&= W_p\left(\beta u(x_1) + (1 - \beta)u(x_3), \beta u(x_2) + (1 - \beta)u(x_4)\right)\\
&= W_p\left(u(F(x_1, x_3; q)), u(F(x_2, x_4; q))\right) = u(F(F(x_1, x_3; q), F(x_2, x_4; q); p))\\
&= u(F(F(x_1, x_2; p), F(x_3, x_4; p); q)) = \beta u(F(x_1, x_2; p)) + (1 - \beta)u(F(x_3, x_4; p))\\
&= \beta W_p(u(x_1), u(x_2)) + (1 - \beta)W_p(u(x_3), u(x_4)).
\end{aligned}$$

Hence, for every $p \in [0, 1]$, the function W_p satisfies functional equation

$$W_p\left(\beta\bar{x} + (1 - \beta)\bar{y}\right) = \beta W_p(\bar{x}) + (1 - \beta)W_p(\bar{y}) \quad \text{for} \quad \bar{x}, \bar{y} \in u(X)^2. \tag{84}$$

Therefore, applying the Daróczy-Páles identity (Daróczy and Páles, 1987)

$$\beta\left(\beta\frac{\bar{x}+\bar{y}}{2}+(1-\beta)\bar{x}\right)+(1-\beta)\left(\beta\bar{y}+(1-\beta)\frac{\bar{x}+\bar{y}}{2}\right)=\frac{\bar{x}+\bar{y}}{2}\quad\text{for}\quad \bar{x},\bar{y}\in u(X)^2,$$

we obtain that, for every $p \in [0,1]$, the function W_p satisfies the two-dimensional Jensen functional equation

$$W_p\left(\frac{\bar{x}+\bar{y}}{2}\right)=\frac{W_p(\bar{x})+W_p(\bar{y})}{2}\quad\text{for}\quad \bar{x},\bar{y}\in u(X)^2,$$

Since X is an interval and u is continuous and strictly increasing, $u(X)^2$ is a non-empty convex subset of $\mathbb{R}^2$. Moreover since F satisfies continuity and monotonicity in each payoff, it is continuous. Thus, for each $p \in [0,1]$, W_p is continuous. Therefore, applying (Kuczma, 2008, Theorem 13.2.2), we conclude that for every $p \in [0,1]$ there exist $w(p), v(p), c(p) \in \mathbb{R}$ such that

$$W_p(x,y)=w(p)x+v(p)y+c(p)\quad\text{for}\quad (x,y)\in u(X)^2. \tag{85}$$

Moreover, making use of **(Ref)**, (83) and (85), for every $p \in [0,1]$ and $x \in u(X)$, we get

$$(w(p)+v(p))x+c(p)=W_p(x,x)=u(F(u^{-1}(x),u^{-1}(x);p))=u(F(u^{-1}(x)))=u(u^{-1}(x))=x. \tag{86}$$

Thus, for every $p \in [0,1]$, we have $w(p)+v(p)=1$ and $c(p)=0$, that is (85) becomes

$$W_p(x,y)=w(p)x+(1-w(p))y\quad\text{for}\quad (x,y)\in u(X)^2. \tag{87}$$

Hence, replacing x and y by $u(x)$ and $u(y)$, respectively, in view of (83), we obtain (15). It follows from **(Ref)**, that $w(0)=0$ and $w(1)=1$. Moreover, as F is monotone with respect to its payoffs, in view of (15), we have $w(p) \in [0,1]$ for $p \in [0,1]$. Thus, since F is strictly increasing in p, w is a probability weighting function. Finally, from (2) and (15) we derive that w is self-conjugate. □

Proof of Theorem 7. That *(i)* implies *(ii)* is straightfoward to verify. We now prove that *(ii)* implies *(i)*. Let

$$M(x,y):=F(x,y;0.5)\quad\text{for}\quad x,y\in X. \tag{88}$$

Then, in view of **(Ref)**, we get

$$M(x,x)=F(x,x;0.5)=F(x)=x\quad\text{for } x\in X,$$
$$M(y,x)=F(y,x;0.5)=F(x,y;0.5)=M(x,y)\quad\text{for}\quad x,y\in X.$$

Thus, M is reflexive and symmetric. Moreover, since F is monotonic in each of its payoffs, M is strictly increasing in each of its variables. So, since F satisfies (19) for $p=q=0.5$, M satisfies

$$M(M(x_1,x_2),M(x_3,x_4))=M(M(x_1,x_3),M(x_2,x_4))$$

for any $(x_1,x_2,x_3,x_4)\in X^4$. Therefore, applying (Burai et al., 2021, Theorem 8) there exists a utility function u such that

$$M(x,y)=u^{-1}\left(\frac{u(x)+u(y)}{2}\right)\quad\text{for}\quad x,y\in X.$$

Hence, by (88), we get

$$F(x,y;0.5) = u^{-1}\left(\frac{u(x)+u(y)}{2}\right) \quad \text{for} \quad x,y \in X. \tag{89}$$

For every $p \in [0,1]$ define a function $W_p : u(X)^2 \to u(X)$ by

$$W_p(s,t) = u(F(u^{-1}(s), u^{-1}(t); p)) \quad \text{for} \quad s,t \in u(X). \tag{90}$$

Then, in view of (89) and *(ii)*, for every $p \in (0,1)$ and $(x_1,x_2,x_3,x_4) \in X^4$ such that $x_1 \geq x_2$ and $x_3 \geq x_4$, we obtain

$$\begin{aligned}
&W_p\left(\tfrac{1}{2}(u(x_1),u(x_2)) + \tfrac{1}{2}(u(x_3),u(x_4))\right) = W_p\left(\tfrac{1}{2}u(x_1) + \tfrac{1}{2}u(x_3), \tfrac{1}{2}u(x_2) + \tfrac{1}{2}u(x_4)\right) \\
&= W_p\left(u(F(x_1,x_3;0.5)), u(F(x_2,x_4;0.5))\right) = u(F(F(x_1,x_3;0.5), F(x_2,x_4;0.5);p)) \\
&= u(F(F(x_1,x_2;p), F(x_3,x_4;p);0.5)) = \tfrac{1}{2}u(F(x_1,x_2;p)) + \tfrac{1}{2}u(F(x_3,x_4;p)) \\
&= \tfrac{1}{2}W_p(u(x_1),u(x_2)) + \tfrac{1}{2}W_p(u(x_3),u(x_4)).
\end{aligned}$$

Hence, for every $p \in [0,1]$, the function W_p satisfies the two-dimensional Jensen functional equation

$$W_p\left(\frac{\bar{x}+\bar{y}}{2}\right) = \frac{W_p(\bar{x}) + W_p(\bar{y})}{2} \quad \text{for} \quad \bar{x},\bar{y} \in D,$$

where $D := \{(u(x),u(y)) : x,y \in X,\ x \geq y\}$. Since X is an interval and u is continuous and strictly increasing, D is a convex subset of $\mathbb{R}^2$. Moreover, since F is monotonic in each payoff and in view of (90), for every $p \in [0,1]$, we have

$$y \leq W_p(x,y) \leq x \quad \text{for} \quad (x,y) \in D.$$

Thus, for every $p \in [0,1]$ and $(x,y) \in D$ with $x > y$, W_p is bounded on $[y,x]^2$ and so, according to (Kuczma, 2008, Lemma 9.3.1, and Theorem 13.2.3), W_p is continuous. Therefore, applying (Kuczma, 2008, Theorem 13.2.2), we conclude that for every $p \in [0,1]$ there exist $w(p), v(p), c(p) \in \mathbb{R}$ such that

$$W_p(x,y) = w(p)x + v(p)y + c(p) \quad \text{for} \quad (x,y) \in D. \tag{91}$$

Moreover, making use of **(Ref)** and (90) and (91), for every $p \in [0,1]$ and $x \in u(X)$, we obtain (86). Thus, for any $p \in [0,1]$, we have $w(p) + v(p) = 1$ and $c(p) = 0$, and so (91) becomes

$$W_p(x,y) = w(p)x + (1-w(p))y \quad \text{for} \quad (x,y) \in D.$$

Hence, replacing x and y by $u(x)$ and $u(y)$, respectively, in view of (90), we get

$$F(x,y;p) = u^{-1}(w(p)u(x) + (1-w(p))u(y)) \quad \text{for} \quad x,y \in X,\ x \geq y, \quad p \in [0,1]. \tag{92}$$

It follows from **(Ref)**, monotonicity of F and (92) that $w(p) \in [0,1]$ for $p \in [0,1]$ and w is non-decreasing with $w(0) = 0$ and $w(1) = 1$. Furthermore, in view of (89) and (92), it follows that $w(0.5) = 0.5$. Finally, the representation for $x < y$ is derived from (2) and (92). □

Proof of Lemma 1. We list the orders in each class

$$\Sigma_1 = \{(1234), (2143), (2413), (4231), (4321), (3412), (3142), (1324)\},$$
$$\Sigma_2 = \{(1243), (1342), (2431), (3421), (4312), (4213), (3124), (2134)\},$$
$$\Sigma_3 = \{(3214), (2314), (1423), (1432), (2341), (3241), (4132), (4123)\}.$$

Note that Σ_1 is a subgroup of Σ generated by two elements: $\tau = (1324)$ and $\rho = (2143)$. So, for $i = 1$ (22) holds. Let $\sigma_2 = (1243)$ and $\sigma_3 = (3214)$. Note that $\sigma_2 \in \Sigma_2$ and $\sigma_3 \in \Sigma_3$. Furthermore, for every $\sigma \in \Sigma_1$, we have

$$((\sigma \circ \sigma_2)(1) - (\sigma \circ \sigma_2)(2))\,((\sigma \circ \sigma_2)(3) - (\sigma \circ \sigma_2)(4)) = (\sigma(1) - \sigma(2))\,(\sigma(4) - \sigma(3)) < 0,$$
$$((\sigma \circ \sigma_2)(1) - (\sigma \circ \sigma_2)(4))\,((\sigma \circ \sigma_2)(2) - (\sigma \circ \sigma_2)(3)) = (\sigma(1) - \sigma(3))\,(\sigma(2) - \sigma(4)) > 0.$$

Hence $\sigma \circ \sigma_2 \in \Sigma_2$ for $\sigma \in \Sigma_1$ and so, as the mapping $\Sigma_1 \ni \sigma \mapsto \sigma \circ \sigma_2 \in \Sigma_2$ is injective and Σ_1 and Σ_2 are finite sets of the same cardinality, we get $\Sigma_1 \circ \sigma_2 = \Sigma_2$. In a similar way, we obtain $\Sigma_1 \circ \sigma_3 = \Sigma_3$. Therefore, for $i \in \{2, 3\}$ and $\sigma \in \Sigma_i$, we have

$$\tau \circ \sigma \in \tau \circ \Sigma_i = \tau \circ \Sigma_1 \circ \sigma_i = \Sigma_1 \circ \sigma_i = \Sigma_i,$$
$$\rho \circ \sigma \in \rho \circ \Sigma_i = \rho \circ \Sigma_1 \circ \sigma_i = \Sigma_i \circ \sigma_i = \Sigma_i.$$

Hence, as τ and ρ generate Σ_1, we obtain $\Sigma_1 \circ \sigma \subset \Sigma_i$ for $\sigma \in \Sigma_i$ and $i \in \{2, 3\}$. Since Σ_i for $i \in \{1, 2, 3\}$ are finite sets of the same cardinality and for any $\sigma \in \Sigma$, a mapping $\Sigma \ni \pi \mapsto \pi \circ \sigma \in \Sigma$ is injective, this yields (22). □

Proof of Lemma 2. Assume that $\Sigma_i(F) \neq \emptyset$ and fix a $\sigma \in \Sigma_i(F)$. Replacing p with q and swapping the sides of (21) yields:

$$F(F(x_1, x_3; p), F(x_2, x_4; p); q) = F(F(x_1, x_2; q), F(x_3, x_4; q); p) \tag{93}$$

for all $p, q \in (0, 1)$ and $(x_1, x_2, x_3, x_4) \in A_\sigma$. Take $\tau = (1324)$ and relabel the variables in (93) by replacing x_j with $x_{\tau(j)}$ for $j \in \{1, 2, 3, 4\}$ to obtain that F satisfies the bisymmetry equation on $A_{\tau\circ\sigma}$, i.e. $\tau \circ \sigma \in \Sigma_i(F)$. Similarly, replacing p with $1 - p$ and applying (2) in (21), we obtain

$$F(F(x_2, x_1; p), F(x_4, x_3; p); q) = F(F(x_2, x_4; q), F(x_1, x_3; q); p) \tag{94}$$

for all $p, q \in (0, 1)$ and $(x_1, x_2, x_3, x_4) \in A_\sigma$. So, taking $\rho = (2143)$ and relabeling the variables in (94) by replacing x_j with $x_{\rho(j)}$, for $j \in \{1, 2, 3, 4\}$, we get that F satisfies the bisymmetry equation on $A_{\rho\circ\sigma}$, i.e. $\rho \circ \sigma \in \Sigma_i(F)$. Hence, as τ and ρ generate Σ_1, we get that $\Sigma_1 \circ \sigma \subset \Sigma_i(F)$. Therefore, applying Lemma 1, we conclude that $\Sigma_i \subset \Sigma_i(F)$. Since by definition $\Sigma_i(F) \subset \Sigma_i$, so $\Sigma_i(F) = \Sigma_i$. □

Proof of Theorem 8. By Corollary 1, *(ii)* implies *(i)* Since w is self-conjugate, F of the form (15) satisfies the bisymmetry equation on X^4. Thus *(i)* implies *(v)*. That *(v)* implies *(iv)* is obvious and, according to Lemma 2, *(iv)* implies *(iii)*. We now prove that *(iii)* implies *(ii)*. First observe that in order to prove $\overline{\textbf{(Dist)}}$ it is sufficient to show that

$$F(F(x, y; p), z; q) = F(F(x, z; q), F(y, z; q); p) \tag{95}$$

holds $p, q \in (0,1)$ and $x, y, z \in X$ with $x \geq y$. Indeed, in such a case replacing p by $1-p$ in (95) and using (2), we obtain that (95) is valid for $p, q \in (0,1)$ and $x, y, z \in X$ with $y \geq x$ as well.

Let $p, q \in (0,1)$ and $x, y, z \in X$ be such that $x \geq y$. Assume that F satisfies the bisymmetry equation on A_σ for some $\sigma \in \Sigma_2$. Hence, according to Lemma 2, $\Sigma_2(F) = \Sigma_2$. As $(1243) \in \Sigma_2$, F satisfies the bisymmetry equation on $A_{(1243)}$, i.e. on $\{(x_1, x_2, x_3, x_4) \in X^4 : x_1 \geq x_2 \geq x_4 \geq x_3\}$. Thus, putting in (21) $x_1 = x$, $x_2 = y$, $x_3 = x_4 = z$, whenever $y \geq z$; and $x_1 = x$, $x_2 = x_4 = z$, $x_3 = y$, whenever $x \geq z \geq y$, we obtain (95). Moreover, as $(3421) \in \Sigma_2$, F satisfies the bisymmetry equation on $A_{(3421)}$, i.e. on $\{(x_1, x_2, x_3, x_4) \in X^4 : x_3 \geq x_4 \geq x_2 \geq x_1\}$. Hence, if $z \geq x$, then setting in (21) $x_1 = y$, $x_2 = x$, $x_3 = x_4 = z$, we get (95) again. □

Proof of Theorem 9. First, we show that

$$\mathcal{A} = \cup_{\sigma \in \Sigma_1} A_\sigma. \tag{96}$$

Note that, if $\sigma \in \Sigma$ and $(x_1, x_2, x_3, x_4) \in A_\sigma$, then for every $i, j \in \{1, 2, 3, 4\}$, we have

$$x_i > x_j \quad \text{if and only if} \quad \sigma^{-1}(i) < \sigma^{-1}(j). \tag{97}$$

Let $(x_1, x_2, x_3, x_4) \in \cup_{\sigma \in \Sigma_1} A_\sigma$. Then $(x_1, x_2, x_3, x_4) \in A_\sigma$ for some $\sigma \in \Sigma_1$. Since Σ_1 is a subgroup of Σ, we have $\sigma^{-1} \in \Sigma_1$ and so, by the definition of Σ_1, we get

$$(\sigma^{-1}(1) - \sigma^{-1}(2))(\sigma^{-1}(3) - \sigma^{-1}(4)) > 0 \text{ and } (\sigma^{-1}(1) - \sigma^{-1}(3))(\sigma^{-1}(2) - \sigma^{-1}(4)) > 0. \tag{98}$$

Thus, applying (97), we obtain $(x_1, x_2, x_3, x_4) \in \mathcal{A}$, which proves that $\cup_{\sigma \in \Sigma_1} A_\sigma \subset \mathcal{A}$. Conversely, assume that $(x_1, x_2, x_3, x_4) \in \mathcal{A}$. Then $(x_1, x_2, x_3, x_4) \in A_\sigma$ for some $\sigma \in \Sigma$. Hence, in view of (97), we get (98), which implies that $\sigma^{-1} \in \Sigma_1$. Using once more the fact that Σ_1 is a subgroup of Σ, we obtain $\sigma \in \Sigma_1$. Thus, we have $(x_1, x_2, x_3, x_4) \in \cup_{\sigma \in \Sigma_1} A_\sigma$. In this way, we have shown that $\mathcal{A} \subset \cup_{\sigma \in \Sigma_1} A_\sigma$. Therefore, (96) holds.

It follows from (96) that *(iv)* implies *(v)* and *(v)* implies *(iii)*. That *(iii)* implies *(iv)* is a consequence of Lemma 2. We now prove that *(iii)* implies *(ii)*. Assume that *(iii)* holds. Then, by the equivalence of *(iii)* and *(iv)*, F satisfies the bisymmetry equation on $A_{(1234)}$, that is on $\{(x_1, x_2, x_3, x_4) \in X^4 : x_1 \geq x_2 \geq x_3 \geq x_4\}$. Thus, taking $p, q \in (0,1)$ and $x, y, z \in X$ with $x \geq y \geq z$ and setting in (21) $x_1 = x$, $x_2 = y$, $x_3 = x_4 = z$, we obtain **(Dist)**. By Theorem 4, *(ii)* implies *(i)*. Finally, it is straighforward to check that F of the form (1) satisfies the bisymmetry equation on $A_{(1234)}$, which shows that *(i)* implies *(iii)*. □

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