Individual and collective rationality in carpooling

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Abstract

We define carpooling as a coalition game and present a socially optimal solution that minimizes the overall cost of commuting and is stable and fair. Instead of transferring costs between players, individual rationality is achieved by appropriate composition and assignment of drivers within carpools. We develop a three-step solution procedure, where our final solution is based on the stable pre-nucleolus of the underlying game. The results of computational experiments show that our procedure guarantees substantial gains from carpooling. These gains increase with the number of commuters and are comparable to gains achieved by centralized systems, which ignore stability and fairness.

1 Introduction

Carpooling allows travelers to share a ride to a common destination, such as a workplace or a school. It provides numerous social and individual benefits. It reduces vehicle miles traveled, fuel consumption, greenhouse gas (GHG) emissions, air pollution (Bruck et al., 2017; Liu et al., 2019), and road congestion (Li et al., 2007; Santi et al., 2014). It saves time, money, and stress (Shaheen et al., 2016) through shared driving responsibilities and travel-time savings associated with High-Occupancy Vehicle (HOV) lane access (Shaheen et al., 2018). It is usually more convenient and flexible than public transport, and it reduces the need for parking space while providing financial and tax benefits for employers (Shaheen et al., 2018).

According to Furuhata et al. (2013) the main challenges to making shared driving more popular are (a) designing attractive mechanisms that encourage participation, (b) coordination of itineraries and schedules, and (c) building trust among unknown passengers. As carpooling involves regular and pre-arranged trips between household members, neighbors, or co-workers, traveler's demand (i.e., travelers' origins/destinations and arrival times) is known beforehand (Mourad et al., 2019; Furuhata et al., 2013). Hence, (b) and (c) are less of a problem than (a). This gives carpooling a comparative advantage over other flexible and *ad hoc* types of ride-sharing and makes it the most popular among them (Morency, 2007). Despite its advantages, the share of drivers using carpooling is relatively low (Olsson et al., 2019) and has declined over the recent years[1].

The main problem seems to be designing attractive carpooling schemes which encourage participation. Due to its voluntary character, it is not sufficient to make carpooling better for the whole society or for a group of commuters. It must be sufficiently better than outside options on an individual level, i.e., for each participant independently. Carpooling should be able to compete with the immediate access to door-to-door transportation that

[1] Of those commuting by automobile, 19.7% carpooled in 1980 as compared to only 9.0% in 2016 (Shaheen et al., 2018).

private cars provide (Agatz et al., 2012). So, a proper design of a carpooling arrangement should maximize the social benefits of carpooling while making it sufficiently better (for each individual) than the outside option of driving alone. In a traditional carpool for recurring trips to work or school, participants typically do not share costs on a per-trip basis but take turns driving their own cars. If the composition of the group of participants differs per trip, which might be the case in the socially optimal carpooling arrangement, it is not trivial to establish a fair driver schedule that respects individual goals (Agatz et al., 2012; Fagin and Williams, 1983; Naor, 2005).

In this paper, we take up this challenge and analyze carpooling from a game-theoretical perspective. We observe that, typically, there is a trade-off between collective and individual rationality. Assigning people to a carpool in a way that provides large overall benefits usually requires driving loads to be allocated highly asymmetrically between the carpool members. One could assume that there exists some efficient means by which drivers may be compensated by riders within a carpool for their extended driving (transferable utility case), but we find this assumption unrealistic. The benefits of a carpool are widely scattered not only among participants but also among those outside of the carpool. On the other hand, the costs of driving one's own car are concentrated on the driver and extend beyond fuel or car depreciation. Time is valuable, and driving requires full attention from the driver[2]. A commuter who is not a professional driver and has other duties and obligations may be unwilling to spend more time driving than when commuting alone. We thus assume, consistent with common practice, that individual rationality (IR) cannot be achieved by compensating participants for their extended driving with money or some other means. Instead, savings are generated by the appropriate composition of carpools and the driver's designation for each carpool so that no individual drives (on average) more than when commuting alone. Not only does this approach allow us to extend existing carpooling models, but it also results

[2]This is important while carpooling to a workplace, but also when driving children to school. In the latter case, only the driver "loses" time, and the other parents save time while their child is being driven in a carpool.

in previously-unknown carpooling arrangements, which increase participation and generate additional savings. We support our findings with experiments on both simulated and real-world instances for which we calculate the savings for each individual and the whole group of commuters in the most common carpooling types.

2 Related literature

Several research papers have been devoted to searching for optimal carpooling arrangements to maximize the benefits of carpooling. Integer linear programs have been proposed to solve the ride-sharing problem in general (Chen et al., 2019) and carpooling as its special case (Baldacci et al., 2004). They propose exact algorithms for problems of limited size and heuristic methods for larger problems.

The majority of papers in the field focus on operational objectives of carpooling, such as optimizing system-wide operating costs. Individual preferences are considered indirectly as constraints in an optimization problem seeking to maximize the social benefits of carpooling. For example, Hasan et al. (2020) analyzes the commute trip-sharing problem to find a routing plan that maximizes ride-sharing while controlling for the ride duration.

However, carpooling is voluntary, and decisions on whether to form a pool are made on an individual level. For this reason, centralized approaches seem insufficient to address individual quality-related objectives. Kalczynski and Miklas-Kalczynska (2019); Miklas-Kalczynska and Kalczynski (2020) consider a decentralized approach and show that it can generate similar system-wide savings as centralized settings while taking carpool members' preferences into account (Mourad et al., 2019, p.329). They consider both the classic to-from carpool type (e.g. commuting to a workplace) as well as less analyzed pickup-dropoff carpool (e.g., driving children to school). The limitation of their approach is that they use a heuristic to capture individual preferences (driving is shared equally among carpool members, savings from carpooling exceed a certain threshold, limited detour distance, etc.). Although the heuristic approach results in substantial savings, their setting does not allow addressing the issue of stability (no incentive to deviate) nor fairness of carpooling arrangements.

Stability requires a game-theoretical framework. It has been studied in many game classes that are related to carpooling. Dreze and Greenberg (1980); Bogomolnaia and Jackson (2002); Aziz et al. (2013); Hasan et al. (2014) analyze the stability of hedonic coalition structures where each player's preferences over partitions of players depend only on the members of their coalitions. Dunne et al. (2010) analyze stability in the context of Coalitional Resource Games (a special case of nontransferable utility games) using a bargaining-based approach to coalition structure generation, and show that a carpool scheme falls within this domain, yet they do not provide the details for this case. Norde et al. (2004) show that carpooling with hubs can be modeled as a minimum cost spanning tree. They assume transferable utility and use the so-called population monotonic allocation scheme, a solution concept in which individual cost does not increase as more people are added to the coalition. Ostrovsky and Schwarz (2019) analyze carpooling as a system that could be implemented using self-driving cars. They take both social and individual objectives into account and show how to ensure stability by setting a toll system. They assume fully transferable utility that is quasilinear in money. Benjaafar et al. (2022) analyze ride-sharing via a platform using a game-theoretic setup. They establish equilibrium conditions by setting up a price system that matches the supply and demand of car rides. Their model uses transferable utility and a continuum of players, and instead of having a specific network structure (players' locations and traffic connections) determine players' strategic position, they assume homogeneous payoffs for all players choosing a given commuting strategy.

The carpooling game considered in this paper does not belong to any of the standard game classes. It is neither the characteristic function game, in which the value of a coalition depends solely on the identities of its members, nor the partition function game, in which the value also depends on how all players, including the nonmember-ones, are partitioned (Rahwan et al., 2015). It is also not a hedonic game because players' preferences depend not only on the carpool to which they belong but also on their share of driving. Unlike in the standard cooperative games, the grand coalition will not form even if the cost is subadditive because the size of a coalition is restricted to $m \leq n$. This feature adds complexity to the problem: one has to search for partitions of the set N into components not to exceed m players (Bényi and Ramírez, 2019).

Any carpooling game is a game in which the solution must be implementable in coalition structures. If utility were fully transferable, it would suffice to restrict coalition shares to be binary, which would lead to a single coalition structure in the optimal solution (a partition of players). This is without loss of generality because a single coalition structure is always in the set of optimal solutions, i.e., it is always optimal for a carpool to choose a driver in such a manner that the distance is minimized. In our game, since we insist on having the individual rationality constraints satisfied with no utility (cost) transfers between players, it is not possible, and we must look for solutions that take the form of a probability distribution over several coalition structures. This approach requires our (more complex) setup.

Our carpooling game falls neither in the pure transferable nor the nontransferable utility class. Individual rationality (no participant drives more on average than when commuting alone) is ensured with no utility (cost) transfers between players. Incentive compatibility requires cost-equivalent transfers between carpool members. When the core is empty (this is possible only in the to-from carpooling type), we determine the amount of cost equivalent that must be added externally to the system to restore stability.

Stability is important to boost carpool participation, yet fairness is also important. Fagin and Williams (1983) and Naor (2005) suggest that a fair-share in a carpool problem is the Shapley value of the carpooling game with transferable utility. The problem with this approach is that, except for convex games, the Shapley value allocation is not guaranteed to be stable, even if the core of a game is nonempty. This is true for carpooling problems, in which the core might be empty. In order to ensure that the solution is stable whenever stable solutions exist, we propose the pre-nucleolus solution (Potters, 1991; Snijders, 1995; Lu and

Quadrifoglio, 2019) rather than the Shapley value solution. The pre-nucleolus also promotes fairness by minimizing the largest "unhappiness" of the coalitions' participants, and it has the advantage of being in the core whenever the core exists. And, if the core doesn't exist, it reveals the amount of cost equivalent that has to be exogenously added to the system to restore stability. This feature constitutes a well-defined and operational policy implication.

3 Statement of the problem

We consider a geographically-dispersed group of commuters who travel to the same destination repetitively using their private vehicles. These could be employees commuting to work on workdays or schoolchildren dropped off at and picked up from school on school days. These commuters can either travel alone or in one of the carpools formed with other commuters; however, they can only be part of a single carpool per trip. Each vehicle owner can act either as a driver or rider (in another vehicle) in a carpool, and the maximum number of available seats restricts the carpool size. Commuters can make different carpooling arrangements for each trip (switch carpools) in such a manner that, each of them will never drive more on average than they would when commuting alone. Such individually rational arrangements could involve a combination of trips for a single commuter: driving alone, driving others in a carpool, and riding as a passenger. The group as a whole will create value from carpooling by reducing the total distance traveled and time spent driving and will thus reduce congestion, pollution, and land use for parking spaces. The policymaker (e.g., school, workplace, local transportation authority) maximizes the value created by such arrangements by defining carpools and specifying the percentage of time each commuter will participate (as a driver or rider) in each of these carpools. The policymaker also determines the fraction of time (if any) each commuter will be driving alone.

While the proposed allocation achieves individual rationality (in a sense defined above) with no transfers between commuters, it does not, in general, ensure stability. While no commuter wants to unilaterally deviate from the proposed scheme, it might be that a group of commuters would form an alternative carpool in which all of its members will be better off. In order for the mechanism to be not only individually-rational but also incentive-compatible and thus stable, we allow limited cost transfers between players. In the second stage of the optimization process, we determine the amounts of transfers between players, which will make the final allocation stable (i.e. in the core). In the baseline case, transfers are equivalent to miles driven. Although this restricts the model (implicitly assuming identical and linear utility functions for each commuter), we show how this assumption can be relaxed. In some cases, the net transfer necessary to restore stability is null, while in others, it is strictly negative. The latter means that the policymaker needs to add an extra amount to the system to make it stable. In such a case, we determine the minimum amount to be added.

Since there might be uncountably many stable allocations in general, and we care about the fairness of allocation as a secondary optimization criterion, our solution will form a stable pre-nucleolus of the associated game, i.e., we will minimize, in the lexicographic order, the greatest utility excess among players, then the second greatest, the third greatest, and so on. It is known that the pre-nucleolus solution always exists, it is unique, belongs to the core whenever it exists, and guarantees fairness in the sense of minimizing the greatest excess. Our approach to designing stable and fair cost allocation scheme for carpooling is similar to that of Lu and Quadrifoglio (2019). Like theirs, our proposed solution is also based on the pre-nucleolus solution. However, unlike them, we do not assume cost transferability among the players and transfers are allowed only in restricted sense.

In this paper, we will provide answers to the following questions:

1. What is the optimal assignment of players and (among them) drivers to carpools so that each player does not drive more on average than when commuting alone, each player is served (i.e., completes the trip at each commuting event), and the overall driving distance is minimized?

2. What are the amount of transfers between players (riders and drivers) so that full

stability and fairness are achieved?

We assume an infinite horizon. Hence a driving arrangement for a player can take the form of a sequence of real-valued non-negative shares adding up to one for the player driving in several different carpools. Yet, the nature of the problem makes it impossible for a player to be a member of two carpools (either as a driver or as a rider) simultaneously. Hence, the solution to the assignment problem must be implementable in coalition structures, i.e., form a probability distribution over partitions of the set of all players. If several coalition structures form, then the total of their shares must equal one (one coalition structure at a time). We investigate what are the potential benefits of switching carpools and carpool structures.

Our approach applies to carpooling and vanpooling: they are both prearranged and typically associated with a daily commute. We distinguish two classes:

a) To/From Carpool (TFC): The vehicle operator picks up riders on the way to a common destination (e.g., a workplace); this is the most common class studied in carpooling literature;

b) Pick Up/Drop Off Carpool (PDC) (Miklas-Kalczynska and Kalczynski, 2020; Kalczynski and Miklas-Kalczynska, 2019): The vehicle operator picks up riders on the way to a common destination (e.g., a school), drops them off, and then returns to the point of origin.

The remainder of the paper is organized as follows. The formal carpooling model and the exact solution to the cost optimization problem are presented in Section 4. Section 5 describes the three-step procedure for finding a fair solution. An algorithm for solving large-scale, practical problems is described in Section 6. Section 7 offers an illustrative example. Section 8 presents the results of computational experiments on simulated and real-world *Daily Carpooling Problem* instances. Section 2 offers a discussion of the results in the context of relevant literature. We conclude the paper in Section 9 and present some future research ideas.

4 The model

Let $N = \{1, 2, ..., n\}$ be a finite set of players (also referred to as participants or players) where $n \geq 2$. Each player commutes to a destination (e.g., a workplace or school), denoted by 0, repeatedly and may do so by either driving alone or by joining a coalition of players (carpool), in which a designated member drives all other members to the destination. The maximum size of a carpool is $m \leq n$ (we thus call the game "multi-restricted," see Choi, 2015). Let $\mathcal{C}_m$ denote the set of possible carpools, i.e., all non-empty subsets of N of size at most m. There are $\sum_{j=1}^{m} \binom{n}{j}$ such coalitions. An m-restricted coalition structure, denoted by P, is a partition (i.e., an exclusive and exhaustive subset) of N containing only sets of at most m players. The set of all such coalition structures is denoted by $\mathcal{P}_m$. The size of $\mathcal{P}_m$ can be determined using a formula in Appendix B. For $C \in \mathcal{C}_m$ and $i \in C$, let $c(C, i)$ denote the cost (distance, time) player i incurs by driving all members of C to a destination. Only the driver incurs the cost of a carpool. To simplify, we denote $c(\{i\}, i)$ by $c(i)$.

We assume that costs $c(C, i)$ are induced by some metric over the space of player locations, i.e., we assume that there exists a space of player possible locations (e.g. $\mathbf{R}^2$) such that $c(C, i)$ is the distance of the driving route in a given carpooling type:

- TFC: the shortest route from i to the destination via all locations in $C \setminus \{i\}$
- PDC: the same as in TFC plus the return distance from 0 to i.

Whether $c(C, i)$ denotes a TFC or a PDC cost will always be given by the context. By this assumption, costs satisfy properties of a metric, in particular the triangle inequality and symmetry. This has implications for both carpooling types (TFC and PDC). For example, there is no loss of generality in considering only half of the path of any carpool in the TFC game, as its distance is always half of the overall distance (see Proposition 2 in Appendix A). In the PDC game, driving together is never worse than driving alone.

We say that two locations i, j exhibit benefits from pooling if the following condition holds: $\min[c(\{i,j\},i), c(\{i,j\},j)] \leq c(i)+c(j)$. Note that in the PDC game this condition holds for any pair of locations. It follows that in this case the optimal partitions in the centralized solution, in which only the overall distance is minimized, belong to the set of coarsest m-restricted partitions (see Proposition 3 in Appendix A). However, in TFC, it might be that pooling i and j together is worse than having them commute separately; to see it consider i and j located on the opposite sides of the destination. In this case driving together implies driving the distance of each player commuting separately plus the distance of getting to one of the payers from the destination.

The profitability of carpooling also depends on the chosen metric. For example, in the TFC game the locations $(1, -0.1)$ and $(1, 1)$, both in $\mathbb{R}^2$ exhibit positive benefits from carpooling under the Euclidean but not under the Manhattan metric.

We are interested in determining a cost allocation scheme, which consists of:

- The driving shares $\delta(C, i) \in [0, 1]$,[3] for each carpool C and $i \in C$
- Cost transfers $t(i)$ to achieve stability/fairness for each player i.

5 The proposed solution

We want to exploit the benefits from centralized carpooling while ensuring stability. Generally, in games like the carpooling game, full stability is very hard or impossible to achieve in the absence of (cost-equivalent) transfers between players. Even worse, there are cases in which extra value has to be added to the system to achieve stability. We now describe our proposed solution which requires only restricted transfers.

[3]Technically, to implement any share in $[0, 1]$, we assume a limit point in a infinitely repeated sequence of commuting events. In practice, only a subset of $[0, 1]$ will be implementable. For example to implement an annual plan involving J school or workdays (typically, 180, or 252, respectively), the implementable shares are contained in the set $\{\frac{i}{J} : i \in \{0, 1, ..., J\}\}$.

We say that a cost allocation scheme $(\delta(C,i), t(i))_{C,i}$ is stable if:

- IR: no player drives more on average than when commuting alone.
- IC: No feasible carpool (including single commuters) may incur lower cost equivalent (driving plus transfers) than under the proposed allocation scheme.

We will find stable cost allocation scheme using a three-step procedure. In the first step we determine the driving shares $\delta(C,i)$ for each carpool and each of its members to minimize overall cost subject to IR constraints. This step does not require any cost-transfers between players. Simply speaking, the first step determines the size of a cake under the IR constraints. This cake will be split more evenly between players in the next two steps. In the second step, for each feasible carpool ($|C| \leq m$) we determine its lowest overall cost in the presence of the IR constraint. In the third step, we will find a stable pre-nucleolus of the game, i.e., we will find cost transfers between players to minimize the maximum (over all feasible carpools) excess between the lowest cost of a carpool found in the second step and the sum of the overall driving cost and transfers for members of this carpool. We will also determine the net transfer (if any) of cost that is necessary for the IC constraints to hold. The proposed solution will thus be stable in the sense defined above. Our proposed solution will also be fair in the sense described above, i.e., we will minimize the maximum excess (Spinetto, 1975). The three steps outlined above are summarized below:

1. Determine a socially optimal solution satisfying Individual Rationality:
   - Assign players to carpools
   - Determine shares of driving in each carpool
   - Determine the overall value
2. Construct incentive compatibility constraints
3. Find the unique stable pre-nucleolus solution

- Determine individual and net transfers to achieve stability/fairness.

We now describe each step in detail.

5.1 Step 1: Socially optimal solution with Individual Rationality

Let $\chi : \mathcal{C}_m \times N \to \{0,1\}$, such that $\chi(C,i) = 1$ if $i \in C$, and $\chi(C,i) = 0$ otherwise. The restriction on the shares of driving δ is the following:

$$0 \leq \delta(C,i) \leq \chi(C,i), \text{ for all } (C,i) \in \mathcal{C}_m \times N, \tag{1}$$

which ensures that the shares are non-negative, do not exceed 1 and are zero for players who do not belong to a given carpool. For carpool C define $\delta(C) := \sum_i \delta(C,i)$, measuring which fraction of commuting events is completed by carpool C. We require that each player tasks are fully completed:

$$\sum_C \delta(C)\chi(C,i) = 1, \text{ for all } i. \tag{2}$$

We also allow switching carpools, i.e., driving or riding in different carpools on different commuting events. By doing this we must rule out that a given commuter is active in two or more different carpools simultaneously. Note that (2) alone is not sufficient. For example let $N = \{1,2,3\}$ and $\delta(\{1,2\}) = \delta(\{1,3\}) = \delta(\{2,3\}) = 0.5$. This solution satisfies (2), yet it requires the players to be simultaneously in two different carpools. On the other hand, the solution $\delta(\{1,2\}) = \delta(\{1,3\}) = \delta(\{3\}) = \delta(\{2\}) = 0.5$ can be implemented with the following two coalition structures each occurring half of the time: $\{\{1,2\},\{3\}\}$ and $\{\{1,3\},\{2\}\}$. We thus impose the following extra restriction. For each coalition structure P let $\theta(P) \geq 0$ be its share. We define $\chi' : \mathcal{P}_m \times \mathcal{C}_m \to \{0,1\}$, where $\chi'(P,C) = 1$ if $C \in P$, and $\chi'(P,C) = 0$ otherwise. We add $\theta(P)$ to the set of controls and require:

$$\sum_P \theta(P)\chi'(P,C) = \delta(C), \text{ for all } C, \tag{3}$$

$$\sum_P \theta(P) = 1. \tag{4}$$

Note that (3) and (4) imply (2) (see Proposition 1 below), but the converse direction does not hold. If costs were fully transferable, the optimal solution would be to assign players to carpools and drivers within each carpool to minimize the overall cost and then (to encourage participation) transfer costs to ensure stability. Here we are interested in the case in which costs are not fully transferable.

Yet, we insist the the allocation scheme be stable. We assume that the players consider the *average* driving cost, not the cost incurred on individual commuting events. In particular, each player refuses to drive more on average than when commuting alone. We thus impose the following restriction:

$$\sum_C \delta(C,i)c(C,i) \leq c(i) \tag{5}$$

And so, our first-stage problem (referred to as M-PART IR) is as follows:

$$\min_{\delta(C,i),\theta(P)} \left\{ \sum_i \sum_C \delta(C,i)c(C,i) \right\}, \tag{6}$$

$$\text{s.t.} \quad 0 \leq \delta(C,i) \leq \chi(C,i), \quad \forall C, i,$$

$$\sum_P \theta(P)\chi'(P,C) = \delta(C), \quad \forall C, \tag{7}$$

$$\sum_P \theta(P) = 1, \quad \theta(P) \geq 0, \quad \forall P, \tag{8}$$

$$\sum_C \delta(C,i)c(C,i) \leq c(i), \quad \forall i. \tag{9}$$

IR (9) holds in a strong sense; namely, it does not require cost transfers between players. Note that, without this constraint, the problem would reduce to a simple assignment task. In such a case it would be optimal to set $\delta(C) = \delta(C, i)$ for $i = \arg\min_{i\in C} c(C, i)$[4]. Setting $c(C) := \min_{i\in C} c(C, i)$, would turn the problem into:

$$\min_{0\leq\delta(C)} \left\{ \sum_i \sum_C \delta(C)c(C) \right\} \tag{10}$$

$$\text{s.t.} \quad \sum_C \delta(C)\chi(C, i) = 1, \quad \text{for each } i, \tag{11}$$

which is equivalent to finding a socially optimal value of the grand coalition in a carpooling game with transferable cost. Ensuring implementability in coalition structures would also be easy in this case, as requiring $\delta(C)$ to be binary combined with (11) forces the solution to form a single coalition structure. Without (9), such a degenerate solution (a single coalition structure played with probability 1) always belongs to the set of optimal solutions. For the next step we need to introduce some additional notation. Let the arguments solving (6) be denoted by δ^* and θ^*. We also define the cost of driving incurred by each driver in the solution as: $c^*(i) := \sum_C \delta^*(C, i)c(C, i)$.

5.2 Step 2: Constructing Incentive Compatibility constraints

As argued in the previous section, when restriction (5) is added, the game cannot be reduced to the characteristic function game. This means that the cost of each carpool depends not only on the identity of its members but also on their driving shares in this coalition. Thus, before we add the IC constraints to ensure stability, we must first determine the lowest cost of each coalition in the presence of the IR constraint (5). In doing this we assume that only carpools (coalitions of size not exceeding m) can make valid objections to the proposed

[4]If $\arg\min_{i\in C} c(C, i)$ is multi-valued, choose exactly one of its elements.

allocation, i.e., for each carpool C we solve the following auxiliary problem:

$$\begin{aligned} & \min_{0 \le \delta(C,i) \le \chi(C,i)} \sum_i \delta(C,i) c(C,i) \\ \text{s.t.} \quad & \sum_i \delta(C,i) = 1, \\ & c(C,i)\delta(C,i) \le c(i), \text{ for each } i \in N. \end{aligned} \tag{12}$$

A solution to this problem always exists in the case of PDC (see Proposition 3, in Appendix A). In this case, since the problems in (12) for each C are fully separable, they can be combined into a single one (call it the IC problem) that gives the same solution:

$$\begin{aligned} & \min_{0 \le \delta(C,i) \le \chi(C,i)} \left\{ \sum_C \sum_i \delta(C,i) c(C,i) \right\} \\ \text{s.t.} \quad & \sum_i \delta(C,i) = 1, \quad \text{for each } C, \\ & c(C,i)\delta(C,i) \le c(i), \quad \text{for each } i \text{ and each } C. \end{aligned} \tag{13}$$

In TFC, in which violations of subadditivity occur, and thus the solution to (13) might not exist, we need to solve the following relaxed optimization problem to find for each carpool the maximum total driving share under the IT constraints:

$$\begin{aligned} & \max_{0 \le \delta(C,i) \le \chi(C,i)} \left\{ \sum_C \sum_i \delta(C,i) \right\} \\ \text{s.t.} \quad & \sum_i \delta(C,i) \le 1, \quad \text{for each } C, \\ & c(C,i)\delta(C,i) \le c(i), \quad \text{for each } i \text{ and each } C. \end{aligned} \tag{14}$$

Let the solution to (14) be denoted by $\delta''(C,i)$. It may be that $\sum_i \delta''(C,i) < 1$. In such cases we replace the $\sum_i \delta(C,i) = 1$ constraint from the original IC problem (13) with $\sum_i \delta(C,i) = \sum_i \delta''(C,i)$. The resulting IC problem is then solved as before.

Denote by $\delta'(C,i)$ the optimal solution to (13) and define the lowest cost satisfying the IR constraint for each coalition as $\bar{c}(C) := \sum_i \delta'(C,i)c(C,i)$. This will be an input to the

Step 3 problem.

5.3 Step 3: Finding the stable pre-nucleolus

If the core exists, the pre-nucleolus belongs to the core. If it doesn't, the prenucleolus is close to the core: in this case we determine the amount of external transfers (or positive net transfers) to ensure full stability. We introduce transfers to achieve incentive compatibility. Note also that, even if individual rationality is fulfilled before transfers are made (5), it may not be the case after transfers are made. So, we need to have the post-transfer IR constraint as well. The transfers are used to reallocate existing resources between players but stability may also require external transfers, which are added to the system. In order to compute the stable pre-nucleolus, we use the procedure introduced in Faigle et al. (2001) involving a sequence of LP problems[5].

Let $t \in \mathbf{R}^n$ denote cost transfers between players. For each carpool we define excess as $e(C,t) = \bar{c}(C) - \sum_{i \in C} (c^*(i) - t(i))$. Here, $c^*(i)$ is the cost of driving incurred by each driver in the IR solution (obtained in Step 1), and $\bar{c}(C)$ is the lowest cost satisfying the IR constraint for each coalition (obtained in Step 2). We then solve:

$$\begin{aligned} & \text{minimize}_{t \in \mathbf{R}^N} z && (15) \\ \text{s.t.} \quad & 0 \leq e(C,t) \leq z, \quad \text{for all } C \in \mathcal{C}_m, \\ & c^*(i) - t(i) \leq c(i) \quad \text{for all } i \in N. \end{aligned}$$

Let z_1 be the optimal value of the above problem. Let $\mathcal{S}_1$ denote the set of all coalitions that

[5]A general scheme for computing the pre-nucleolus in a form of a sequential linear program was first introduced in (Maschler et al., 1979).

are binding in the corresponding optimal solution. Then we solve the following problem:

$$\text{minimize}_{t\in\mathbf{R}^N, z\in\mathbf{R}} \{z\} \tag{16}$$

$$\text{s.t.} \quad e(C,t) = z_1, \quad \text{for all } C \in \mathcal{S}_1$$

$$0 \le e(C,t) \le z, \quad \text{otherwise}$$

$$c^*(i) - t(i) \le c(i) \quad \text{for all } i \in N.$$

Let z_2 be the optimal value of the above problem. Let $\mathcal{S}_2$ denote the set of all coalitions that are binding in the corresponding optimal solution. We then continue by solving for i:

$$\text{minimize}_{t\in\mathbf{R}^N, z\in\mathbf{R}} \{z\} \tag{17}$$

$$\text{s.t.} \quad e(C,t) = z_i, \quad \text{for all } C \in \mathcal{S}_i, \quad i \in \{1, 2, ..., i-1\},$$

$$0 \le e(C,t) \le z, \quad \text{otherwise}$$

$$c^*(i) - t(i) \le c(i) \quad \text{for all } i \in N.$$

until all constraints are binding at some step. As a result of this sequential procedure we obtain a sequence $z_1 > z_2 > ... > z_k$. The optimal solution is unique and forms the stable pre-nucleolus of the carpooling game.

In the theoretical worst-case scenario, the length of the sequence, i.e., the number of LPs to be solved in Step 3 will be equal to the number of coalitions in $\mathcal{C}_m$. In practice, however, the length of the sequence is much smaller and did not exceed 19 in all our experiments.

6 The Solution Algorithm to Step 1

Note that the optimization problems outlined in steps 2 and 3 are computationally less demanding than that in step 1, which requires listing all partitions $\mathcal{P}_m$ ($m \le n$). Therefore, we define two simpler problems and prove that they form tight lower and upper bounds on the original problem. These two problems require the enumeration of coalitions but not partitions. Recall, that for both the TFC and PDC models, we consider only the shortest path. In the case of passenger cars, this involves finding the shortest path among a small number of locations (usually, up to 6). This makes the total enumeration of coalitions possible for practical problems with more than a hundred commuters.

6.1 Tight lower and upper bounds

The first problem differs from the M-PART IR in that it does not require the solution to form a coalition structure (partition). It replaces (3) and (4) with (2). It is an LP problem with $|N| \times |\mathcal{C}_m|$ decision variables.

NO-PART IR

$$\min_{\delta(C,i)} \left\{ \sum_i \sum_C \delta(C,i) c(C,i) \right\}, \tag{18}$$

$$\text{s.t.} \quad 0 \leq \delta(C,i) \leq \chi(C,i), \quad \forall C, i,$$

$$\sum_C \delta(C)\chi(C,i) = 1, \quad \forall i, \tag{19}$$

$$\sum_C \delta(C,i) c(C,i) \leq c(i), \quad \forall i.$$

The second problem requires the solution to form a partition, but instead of allowing all possible partitions in $\mathcal{P}_m$, it will restrict the solution to be implementable and optimal in the class of single coalition structure problems.

1-PART IR

$$\min_{\delta(C,i),\gamma(C)} \left\{ \sum_i \sum_C \delta(C,i) c(C,i) \right\}, \tag{20}$$

$$\text{s.t.} \quad 0 \leq \delta(C,i) \leq \chi(C,i), \quad \forall C, i.$$

$$\sum_C \delta(C)\chi(C,i) = 1, \quad \forall i, \tag{21}$$

$$\gamma(C) = \delta(C), \quad \forall C, \tag{22}$$

$$\gamma(C) \in \{0,1\}, \quad \forall C, \tag{23}$$

$$\sum_C \delta(C,i) c(C,i) \leq c(i), \quad \forall i.$$

Similarly to problem (10), partition-implementability in the above problem is ensured by requiring the decision variable $\gamma(C)$, or $\delta(C)$ in (10), to be binary. The difference is that in the presence of the IR constraints (5), the single partition solution may not be optimal in the class of all partition-implementable solutions, but only in those that contain a single partition. This leads to the following proposition, the proof of which is given in Appendix A.

Proposition 1 (Lower and upper bounds)**.** *The optimal overall cost of M-PART IR is no greater than that of 1-PART IR and no smaller than that of NO-PART IR. Moreover, these bounds are tight, i.e., there are instances such that the optimal cost in M-PART IR is equal to that in 1-PART IR and/or equal to that in NO-PART IR.*

6.2 The algorithm

In general, our model requires the total enumeration of multi-restricted coalitions and partitions with the size of the latter approaching the Bell B number (B_n) (see, e.g., Bényi and Ramírez (2019)) for the number of players. To make the model practical, we propose an effective algorithm that can be used to obtain a solution for larger problems, for which all coalitions (but not partitions) can be enumerated. This approach will work even if a subset of all possible coalitions is selected.

1. Find the NO-PART IR solution and determine the coalitions in the solution.
2. Use *Algorithm X* (Knuth, 1999) to partition the coalitions.
3. If *Algorithm X* terminates successfully, i.e., finds a partitioning of the set of coalitions, go to Step 5.
4. Otherwise,i.e., when partitioning of the set is not possible, solve the 1-PART IR problem, unionize the coalitions from the solution with those from Step 1 and apply *Algorithm X* again.
5. If only one partition is formed, then it is the solution. STOP.
6. Otherwise, solve the M-PART IR model using the partitions formed by *Algorithm X* and the corresponding coalitions as input.

Although *Algorithm X*'s worst-case computational complexity is exponential, such cases almost never occur in practical applications. Note that, if *Algorithm X* terminates successfully in Step 3 (and all coalitions are considered), the final solution is optimal by Proposition 1. Otherwise, the solution is a heuristic solution (upper bound on the minimum cost). In addition, since by Proposition 1 the NO-PART IR solution is a lower bound on the minimum cost, managers can determine the quality of the heuristic solution by comparing it to the bound. The reader is referred to Appendix B for suggested techniques for selecting candidate pools and partitions.

7 An illustrative example

Here, we illustrate the solution approach and its intuitions using a simple example. Consider a TFC carpooling type with $n = 12$ and $m = 4$, i.e., 12 commuters driving to work daily, each car having 4 seats. The commuters' locations are depicted in Figure 1 and the destination is marked with a circle. The goal is to minimize the total Euclidean distance traveled by

Figure 1: Location map for the illustrative example

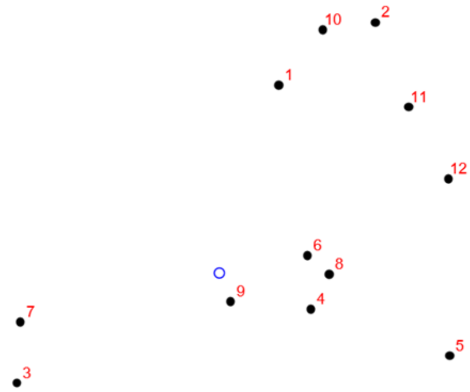

Table 1: Driving distances in various solutions

Commuter	1	2	3	4	5	6	7	8	9	10	11	12	Total
Base	0.78	1.17	0.91	0.39	0.97	0.35	0.81	0.43	0.12	1.05	1	0.98	8.98
NO-PART IR	0	0.6	0.91	0	0.97	0	0.17	0	0.09	0.38	0.62	0.98	4.72
1-PART IR	0	1.17	0.91	0.08	0.97	0	0.15	0	0	0.12	1	0.44	4.84
M-PART IR	0	1.17	0.91	0.02	0.97	0	0.15	0	0	0	1	0.58	4.81
PART NO-IR	0	1.27	1.05	0	1.02	0	0	0	0	0	1.4	0	4.75

all commuters. Table 1 shows players' driving distances for different solutions. The base distance is 8.98, which is the distance traveled when everyone commutes alone. PART NO-IR is the centralized solution (10), in which the following carpools are formed (drivers are emphasized): $\{\mathbf{3}, 7\}, \{1, \mathbf{2}, 10\}, \{4, \mathbf{5}, 9\}, \{6, 8, \mathbf{11}, 12\}$. The overall distance is 4.75, and the difference from the base, i.e., 4.23 is the potential reduction in vehicle miles traveled. Yet, if costs are not fully transferable, this centralized solution is not stable. In particular, the drivers (commuters $2, 3, 5, 11$) drive on average more than without carpooling.

One way to ensure individual rationality is to rotate the drivers in each carpool. This is achieved in the 1-PART IR solution, where the optimal partition remains the same but the driving responsibility is shared shared among the carpool members (see Table 2 for the

Table 2: Driving shares in the 1-PART solution

Carpool	1	2	3	4	5	6	7	8	9	10	11	12	Sum
3, **7**			0.87				0.13						1
1, **2**, **10**		0.92								0.08			1
4, **5**, 9				0.05	0.95								1
6, 8, **11**, **12**											0.71	0.29	1

Table 3: Driving shares in the M-PART solution

Carpool	1	2	3	4	5	6	7	8	9	10	11	12	Sum
3, **7**			0.87				0.13						1.00
4, 9				0.05									0.05
1, **2**, 10		0.92											0.92
4, **5**, 9					0.95								0.95
6, 8, **12**												0.03	0.03
1, 2, 10, **11**											0.08		0.08
5, 6, 8, **12**												0.05	0.05
6, 8, **11**, **12**											0.62	0.3	0.92

Note: Drivers are marked in bold

Table 4: Partitions and their shares in the M-PART IR solution

Partitions				Partition share
3, **7**	1, **2**, 10	4, **5**, 9	6, 8, **11**, **12**	0.9201
3, **7**	1, 2, 10, **11**	**4**, 9	5, 6, 8, **12**	0.0514
3, **7**	1, 2, 10, **11**	4, **5**, 9	6, 8, **12**	0.0285

solution). In this solution players cannot change carpools (there is only one partition). One can improve this solution by allowing such possibility: this is the case of the M-PART IR solution presented in Table 3, where multiple partitions are allowed. For example commuter 11 acts as a driver in carpool $\{1, 2, 10, 11\}$ (share of 0.08) and both as a driver (share of 0.62) and as a rider (share of 0.3) in carpool $\{6, 8, 11, 12\}$.

Here, the solution to M-PART IR is exact, yet due to the problem complexity (the number of all pools increases exponentially – and the number of partitions faster than exponentially – with the number of commuters) larger problems often allow only for approximations. By Proposition 1, 1-PART IR and NO-PART IR are, respectively, the upper and lower bounds. In the example considered, the objective of 1-PART IR is only slightly higher (4.84) than that of M-PART IR (4.81).

In both M-PART and 1-PART, the solutions have to be implementable in coalition structures. Table 4 shows how the individual carpool shares translate into partition shares for the M-PART IR case. On the other hand, NO-PART IR does not require carpools to form partitions. In this case, IR are ensured only by switching carpool membership, but within each carpool only a single commuter acts as a driver. However, this is not a valid solution

Table 5: The stable pre-nucleolus solution: transfers and cost after transfers

	1	2	3	4	5	6	7	8	9	10	11	12	Sum
Cost	0	1.17	0.91	0.02	0.97	0	0.15	0	0	0	1	0.58	4.81
Transfer	0.34	-0.65	-0.31	0.16	-0.18	0.11	0.31	0.19	-0.09	0.43	-0.59	-0.04	-0.32
Net cost	0.34	0.52	0.6	0.18	0.79	0.11	0.46	0.19	-0.09	0.43	0.41	0.54	4.49

because it might require a single commuter to act as a driver and a rider in two different carpools at the same time, i.e., in a single commuting event.

The second step of the solution algorithm involves constructing incentive compatibility constraints. We thus solve the IC problem (13) with further modifications for TFC in order to get the minimum cost in the presence of the IR constraint $\bar{c}(C)$ for each carpool. For example, it is 1.068 for carpool $\{3, 7\}$ (the same as the total driving cost of 3 and 7 in the M-PART and 1-PART solution), and 2.868 for carpool $\{1, 2, 3\}$, which results from splitting driving among all three members. In what follows, we do not explicitly present the results of this auxiliary intermediate step.

After completing the second step, we compute the stable pre-nucleolus and report its solution in Table 5. "Cost" in the first row corresponds to the driving distance from the M-PART IR solution. Even though IR is satisfied and nobody drives more than with no carpooling, the driving cost is distributed unevenly among commuters, and it does not reflect their strategic positions. For example, commuters $1, 6, 8, 9, 10$ do not drive at all, while $2, 3, 5, 11$ drive as much as when commuting alone. In order to restore full stability and fairness (in the sense of minimizing the largest excess), it is necessary to introduce cost transfers. In this case it is not sufficient to redistribute existing costs among commuters while keeping the sum of cost equivalent constant, but rather one has to add extra transfers from the outside of the system. This is reflected in the sum of transfers being negative, which determines the cost of achieving a stable and fair solution. Costs are redistributed so that driving is spread among the commuters. This spread is far from even because it reflects the strategic position of each commuter. Indeed, the net cost per each of the commuters: $2, 3, 5, 12$ is much higher than that of commuters: $4, 6, 8, 9$. This is because the commuters in the former group are located farther from the destination than those from the latter group. But this is not always the case because the cost after transfer of commuter 2 is lower than that of commuter 5, even though 2 is located farther from the destination. This, in turn, is due to the strategic positions of 2 and 5: in the optimal solution 2 always pools with $1, 10$ (and sometimes also with 11), while 5 pools mostly with 4 and 9. Such an arrangement is natural because their respective partners are close to the line connecting them with the destination or "on the way." However, while 1 and 10 have a long way to travel (just like 2, 4 and 9), they are located much closer to the destination as compared to 5. It means that the strategic position of 5 is weaker than that of 2, who can share driving more evenly with other carpool members. Note that the position of 9 is so strong that its net cost is negative. This is so because 9 does not need anyone to pool with whereas other participants need 9.

8 Computational Experiments

All experiments were run on a virtualized Linux Ubuntu 20.04 LTS environment with 38 vCPUs and 374GB of vRAM. The physical server used was a VxRail V570F with Intel Xeon Gold 6248 @ 2.5GHz (2 sockets, 20 cores per processor, 80 logical processors), 748GB RAM, and VMware vSAN storage. Mathematica 13.0 was used for mathematical modeling and generating sparse matrices and vectors for linear programs. All linear programs were solved with Gurobi 9.5. Matlab R2021b was used to control Gurobi.

In order to validate the proposed solution approach, we first tested it on a set of smaller problems, for which (the optional) total enumeration of pools and partitions is possible in reasonable time.

To this end, we randomly generated 35 instances with $n = 12$, $m = 4$, i.e., with 793 carpools and $3,305,017$ set partitions. The coordinates of these simulated instances are available in our Open Science Framework repository at https://osf.io/z9fb3/. The goal is to minimize the total travel distance, assumed to be Euclidean.

Tables 6-9 present the results of numerical experiments on the simulated instances. All enumerations in these tables are total and all solutions are optimal. The instance identifier is in the first column, and *Base* shows the total travel distance without carpooling. M-PART IR is the Step 1 solution objective, while NO-PART IR and 1-PART IR are its lower and upper bounds. The number of set partitions in the M-PART IR solution is denoted by p. *Sav.* are the relative savings as compared to *Base*, *Gap* is the relative difference between the M-PART IR and the NO-PART IR (LB) objectives, and *M vs. 1* is the percentage point gain in relative savings due to considering multiple partitions as compared to a single partition solution.

Tables 6 and 7 show the results of Step 1 of our optimization procedure for TFC and the PDC carpooling models respectively. Note that all solutions to the simulated instances are optimal because we consider all carpools and partitions. The savings were generally higher for the PDC case with the average of 54.6%, compared to 41.89% for the TFC case. Bounds established by Proposition 1 are tight. This is confirmed by the optimal cost (M-PART IR) being equal to the cost of LB (NO-PART IR) for some cases and equal to the cost of UB (1-PART IR) in some other cases. In fact, there were instances in which:

- the three solutions coincided (indicated by a 0% gap, e.g., TFC: R4, PDC: R2)
- M-PART coincided with the UB, but not with LB (e.g., TFC: R13, PDC: R1)
- M-PART coincided with the LB, but not with the UB (e.g., TFC: R14, PDC: R3)
- all three solutions differed (e.g., TFC: R2, PDC: R4).

This is shown in the tables by emphasizing the coinciding solutions. The average time to solve a single NO-PART IR, 1-PART IR, and M-PART IR TFC (PDC) problem was 0.02 (0.02), 0.11 (0.29), and 514 (466) seconds respectively. The fact that 1-PART is generally solved more than 100 times faster than M-PART, and that the loss of savings due to considering one instead of multiple partitions is rather low (the maximum out of all instances was 1.8 pp.

Table 6: The individually rational (Step 1) solutions to simulated carpooling instances – TFC

Inst.	Base	NO-PART IR (LB)	1-PART IR (UB)	M-PART IR	p	Sav. [%]	Gap [%]	M vs. 1 [pp]
R1	9.147	5.644	5.662	5.644	4	38.30	0.01	0.20
R2	8.145	5.195	5.234	5.203	3	36.12	0.15	0.38
R3	9.024	5.780	5.818	5.780	3	35.95	0.00	0.42
R4	8.333	**5.220**	**5.220**	**5.220**	1	37.35	0.00	0.00
R5	12.089	6.754	7.241	7.098	4	41.29	5.09	1.19
R6	10.803	**6.296**	**6.296**	**6.296**	1	41.72	0.00	0.00
R7	10.149	5.486	5.528	5.490	3	45.90	0.07	0.37
R8	10.344	6.197	6.323	6.199	3	40.07	0.04	1.20
R9	9.045	4.359	4.389	4.387	2	51.50	0.65	0.02
R10	9.382	5.438	5.623	5.561	3	40.73	2.25	0.67
R11	9.373	4.710	4.821	4.820	2	48.58	2.34	0.01
R12	8.946	**5.333**	**5.333**	**5.333**	1	40.39	0.00	0.00
R13	11.165	5.949	**6.047**	**6.047**	1	45.84	1.65	0.00
R14	7.645	**4.410**	4.411	**4.410**	2	42.32	0.00	0.01
R15	9.088	5.587	5.640	5.594	2	38.44	0.12	0.51
R16	10.093	5.839	5.915	5.847	3	42.07	0.14	0.68
R17	8.556	**4.368**	**4.368**	**4.368**	1	48.95	0.00	0.00
R18	8.669	5.318	5.447	5.348	3	38.31	0.56	1.14
R19	9.214	5.647	5.828	5.662	5	38.56	0.27	1.80
R20	8.975	4.721	4.843	4.809	3	46.42	1.85	0.38
R21	7.654	**4.612**	4.652	**4.612**	3	39.75	0.00	0.53
R22	9.852	**5.401**	**5.401**	**5.401**	2	45.18	0.00	0.01
R23	8.588	**5.133**	5.184	**5.133**	3	40.23	0.00	0.59
R24	10.661	5.274	**5.314**	**5.314**	1	50.16	0.76	0.00
R25	9.757	4.660	**5.097**	**5.097**	1	47.76	9.37	0.00
R26	10.054	**5.991**	6.043	**5.991**	2	40.41	0.00	0.52
R27	9.479	5.298	5.368	5.335	3	43.71	0.70	0.34
R28	9.371	6.018	6.205	6.058	6	35.35	0.68	1.56
R29	6.918	3.927	3.984	3.981	2	42.45	1.39	0.03
R30	9.655	5.626	5.653	5.626	3	41.73	0.01	0.28
R31	10.502	6.029	6.067	6.061	3	42.28	0.54	0.05
R32	9.823	5.726	5.902	5.896	3	39.98	2.98	0.05
R33	7.868	4.867	4.895	4.889	3	37.86	0.45	0.07
R34	8.389	**4.995**	5.076	**4.995**	4	40.46	0.00	0.97
R35	7.038	4.190	4.254	4.209	3	40.19	0.46	0.64

Optimal solutions which coincide with the NO-PART IR or 1-PART IR solutions are emphasized.

for TFC and 0.7 pp. for PDC). This suggests that 1-PART solutions might be sufficiently good for larger practical applications. They are considerably easier to implement as each commuter is assigned to only a single carpool and savings are comparable to the M-PART solution.

Table 7: The individually rational (Step 1) solutions to simulated carpooling instances – PDC

Inst.	**Base**	**NO-PART IR (LB)**	**1-PART IR (UB)**	**M-PART IR**	p	**Sav. [%]**	**Gap [%]**	**M vs. 1 [pp]**
R1	18.294	8.653	**8.654**	**8.654**	1	52.70	0.00	0.00
R2	16.289	**7.961**	**7.961**	**7.961**	1	51.13	0.00	0.00
R3	18.048	**8.890**	8.966	**8.890**	2	50.74	0.00	0.42
R4	16.666	7.789	8.074	7.992	2	52.05	2.60	0.49
R5	24.178	10.483	10.649	10.600	2	56.16	1.11	0.20
R6	21.606	9.666	9.738	9.683	3	55.18	0.18	0.25
R7	20.297	**8.866**	**8.866**	**8.866**	1	56.32	0.00	0.00
R8	20.688	**8.879**	**8.879**	**8.879**	1	57.08	0.00	0.00
R9	18.090	**7.202**	**7.202**	**7.202**	1	60.19	0.00	0.00
R10	18.764	8.461	8.702	8.685	2	53.71	2.66	0.09
R11	18.747	7.547	7.704	7.682	2	59.02	1.79	0.12
R12	17.892	8.486	8.493	8.488	2	52.56	0.02	0.03
R13	22.329	9.339	9.687	9.639	2	56.83	3.20	0.22
R14	15.291	6.967	7.238	7.130	4	53.37	2.34	0.70
R15	18.175	**8.370**	**8.370**	**8.370**	1	53.95	0.00	0.00
R16	20.185	9.449	9.815	9.789	2	51.50	3.60	0.13
R17	17.111	**6.999**	**6.999**	**6.999**	1	59.10	0.00	0.00
R18	17.337	7.556	**7.558**	**7.558**	1	56.41	0.03	0.00
R19	18.429	8.525	8.687	8.652	2	53.05	1.48	0.19
R20	17.950	7.672	**8.017**	**8.017**	1	55.34	4.50	0.00
R21	15.309	**7.127**	**7.127**	**7.127**	1	53.45	0.00	0.00
R22	19.704	**7.987**	**7.987**	**7.987**	1	59.46	0.00	0.00
R23	17.176	7.787	**7.797**	**7.797**	1	54.60	0.13	0.00
R24	21.321	8.959	**9.251**	**9.251**	1	56.61	3.25	0.00
R25	19.513	8.079	**8.269**	**8.269**	1	57.62	2.35	0.00
R26	20.109	**9.039**	**9.039**	**9.039**	1	55.05	0.00	0.00
R27	18.958	8.610	8.719	8.717	2	54.02	1.25	0.01
R28	18.743	**8.891**	8.986	**8.891**	2	52.56	0.00	0.50
R29	13.836	**6.410**	**6.410**	**6.410**	1	53.67	0.00	0.00
R30	19.311	8.886	9.132	9.011	2	53.34	1.40	0.63
R31	21.003	9.817	9.971	9.970	2	52.53	1.56	0.00
R32	19.647	9.264	9.426	9.378	2	52.27	1.23	0.25
R33	15.735	7.737	7.859	7.782	2	50.54	0.58	0.49
R34	16.778	**7.522**	7.554	**7.522**	2	55.17	0.00	0.19
R35	14.077	6.378	6.566	6.503	3	53.80	1.96	0.45

Optimal solutions which coincide with the NO-PART IR or 1-PART IR solutions are emphasized.

Tables 8 and 9 summarize the results of Step 3 of our optimization procedure for the TFC and PDC carpools respectively. They show the *Base* and Step 1 solution objective

Table 8: The stable pre-nucleolus (Step 3) solutions to the simulated instances – TFC

Inst.	**Base**	**PART NO IR**	**M-PART IR**	**Net Tr.**	**Max Tr.**	**Min Tr.**	**Loss IR [pp.]**	**Cost of IC [pp.]**
R1	9.147	5.512	5.644	-0.286	0.498	-0.421	1.44	3.13
R2	8.145	5.040	5.203	-0.343	0.414	-0.456	2.00	4.21
R3	9.024	5.673	5.780	-0.402	0.336	-0.306	1.19	4.46
R4	8.333	5.063	5.220	-0.000	0.271	-0.450	1.89	0.00
R5	12.089	6.875	7.098	-0.517	0.630	-0.769	1.84	4.27
R6	10.803	6.207	6.296	-0.000	0.639	-0.528	0.82	0.00
R7	10.149	5.352	5.490	-0.161	0.317	-0.739	1.37	1.58
R8	10.344	6.063	6.199	-0.010	0.663	-0.552	1.32	0.09
R9	9.045	4.340	4.387	-0.656	0.478	-0.519	0.52	7.26
R10	9.382	5.435	5.561	-0.596	0.545	-0.624	1.34	6.36
R11	9.373	4.799	4.820	-0.564	0.415	-0.465	0.22	6.02
R12	8.946	5.198	5.333	-0.047	0.498	-0.623	1.51	0.53
R13	11.165	5.954	6.047	-1.063	0.502	-0.549	0.83	9.52
R14	7.645	4.263	4.410	-0.320	0.331	-0.421	1.92	4.18
R15	9.088	5.375	5.594	-0.137	0.472	-0.542	2.41	1.51
R16	10.093	5.745	5.847	-0.262	0.276	-0.587	1.01	2.60
R17	8.556	4.190	4.368	-0.328	0.430	-0.457	2.08	3.83
R18	8.669	5.248	5.348	-0.412	0.614	-0.388	1.15	4.75
R19	9.214	5.465	5.662	-0.000	0.534	-0.504	2.14	0.00
R20	8.975	4.748	4.809	-0.316	0.429	-0.651	0.67	3.52
R21	7.654	4.378	4.612	-0.112	0.334	-0.378	3.06	1.46
R22	9.852	5.353	5.401	-0.957	0.463	-0.502	0.49	9.72
R23	8.588	5.044	5.133	-0.259	0.517	-0.498	1.04	3.01
R24	10.661	5.238	5.314	-0.144	0.426	-0.623	0.71	1.35
R25	9.757	4.902	5.097	-0.751	0.395	-0.619	2.00	7.69
R26	10.054	5.775	5.991	-0.245	0.406	-0.543	2.15	2.44
R27	9.479	5.192	5.335	-0.506	0.450	-0.651	1.52	5.33
R28	9.371	5.932	6.058	-0.077	0.521	-0.451	1.35	0.82
R29	6.918	3.894	3.981	-0.353	0.211	-0.410	1.26	5.10
R30	9.655	5.533	5.626	-0.954	0.603	-0.613	0.97	9.88
R31	10.502	5.968	6.061	-0.654	0.440	-0.705	0.89	6.22
R32	9.823	5.793	5.896	-0.274	0.416	-0.715	1.05	2.79
R33	7.868	4.680	4.889	-0.498	0.504	-0.336	2.66	6.33
R34	8.389	4.897	4.995	-0.156	0.462	-0.474	1.17	1.86
R35	7.038	4.116	4.209	-0.091	0.287	-0.361	1.33	1.30

(*M-PART IR*). *PART NO IR* is the centralized solution objective, which does not require IR. It is compared to the decentralized solution (M-PART IR) in the *Loss IR* column, which gives the loss of savings (relative to *Base*) due to IR. *Net Tr.* gives the net cost equivalent needed to be exogenously added to the system to ensure stability (IC and IR after transfers). *Max Tr.* and *Min Tr.* are, respectively, the maximum and the minimum individual transfer of cost equivalent. Finally, *Cost of IC* is the ratio of what has to be added to the system to ensure stability (i.e., the negative of net transfers) to the *Base* solution objective.

Meeting the incentive compatibility constraints turns out to be costly in both TFC and PDC carpooling types. This is reflected in net transfers being strictly negative for most of the instances. Only three instances in the TFC and two in PDC did not incur additional cost. Even if the net transfer is zero, meaning that the core is nonempty and stability does not require an outside cost, some transfers between commuters are generally needed to ensure stability. This is for example the case for R4 in TFC carpooling, in which net transfer is zero, yet the minimum transfer is -0.450 and the maximum transfer is 0.271.

The loss due to IR can be directly compared to the cost of IC, as both are shown relative to the base. The mean value of *Loss IR* (3.8 pp. for TFC and 2.46 pp. for PDC) was substantially larger than the mean value of the *Cost of IC* (1.41 pp. for TFC and 0.28 pp. for PDC). It suggests that IC is, in general, more costly than IR. However, there are cases in which the opposite is true (e.g. R21 for TFC and R18 for PDC).

9 Discussion and concluding remarks

In this paper, we define carpooling as a new class of coalition games. We provide a three-step procedure to obtain a stable solution to the carpooling game. In the first step, we determine socially-optimal coalition structures, which also satisfy the individual rationality constraints. Our solution achieves individual rationality in the long run with no utility (cost) transfers, but only by appropriate switching of carpool membership and of driving vs. riding roles within each carpool. Cost transfers are added only at a later stage to ensure full stability. We construct incentive compatibility constraints in the second step and, finally, obtain a

Table 9: The stable pre-nucleolus (Step 3) solutions to the simulated instances – PDC

Inst.	**Base**	**PART NO IR**	**M-PART IR**	**Net Tr.**	**Max Tr.**	**Min Tr.**	**Loss IR [pp.]**	**Cost of IC [pp.]**
R1	18.294	8.650	8.654	-0.172	0.801	-0.952	0.02	0.94
R2	16.289	7.842	7.961	-0.019	0.870	-1.042	0.73	0.12
R3	18.048	8.799	8.890	-0.016	1.150	-0.916	0.51	0.09
R4	16.666	7.987	7.992	-0.689	0.916	-1.098	0.03	4.13
R5	24.178	10.495	10.600	-0.561	1.098	-1.511	0.43	2.32
R6	21.606	9.602	9.683	-0.280	1.285	-1.410	0.37	1.30
R7	20.297	8.866	8.866	-0.545	0.941	-1.589	0.00	2.69
R8	20.688	8.834	8.879	-0.064	1.547	-1.332	0.22	0.31
R9	18.090	7.187	7.202	-0.217	1.011	-1.177	0.08	1.20
R10	18.764	8.588	8.685	-0.544	1.269	-1.225	0.52	2.90
R11	18.747	7.675	7.682	-0.453	1.312	-1.105	0.04	2.42
R12	17.892	8.465	8.488	-0.806	1.098	-0.949	0.13	4.51
R13	22.329	9.533	9.639	-0.638	1.390	-1.192	0.47	2.86
R14	15.291	7.114	7.130	-0.341	1.289	-0.919	0.10	2.23
R15	18.175	8.342	8.370	-0.162	1.031	-1.087	0.16	0.89
R16	20.185	9.628	9.789	-0.921	1.769	-1.261	0.80	4.56
R17	17.111	6.991	6.999	-0.005	0.932	-0.921	0.05	0.03
R18	17.337	7.492	7.558	0.000	1.154	-1.143	0.38	0.00
R19	18.429	8.650	8.652	-0.841	1.181	-0.923	0.01	4.56
R20	17.950	8.004	8.017	-0.571	1.208	-1.182	0.07	3.18
R21	15.309	7.091	7.127	-0.129	1.253	-0.953	0.23	0.84
R22	19.704	7.959	7.987	0.000	1.638	-1.078	0.14	0.00
R23	17.176	7.717	7.797	-0.041	1.160	-1.357	0.47	0.24
R24	21.321	9.249	9.251	-1.386	1.197	-1.401	0.01	6.50
R25	19.513	8.225	8.269	-0.683	1.366	-1.309	0.23	3.50
R26	20.109	9.038	9.039	-1.058	1.100	-1.459	0.00	5.26
R27	18.958	8.661	8.717	-1.418	1.168	-1.477	0.30	7.48
R28	18.743	8.862	8.891	-0.397	0.977	-1.018	0.16	2.12
R29	13.836	6.254	6.410	-0.141	1.071	-0.874	1.13	1.02
R30	19.311	8.946	9.011	-0.940	1.385	-0.824	0.34	4.87
R31	21.003	9.931	9.970	-0.752	1.338	-1.346	0.19	3.58
R32	19.647	9.208	9.378	-0.843	0.964	-1.382	0.86	4.29
R33	15.735	7.749	7.782	-0.195	1.538	-0.836	0.21	1.24
R34	16.778	7.496	7.522	-0.245	1.025	-1.047	0.16	1.46
R35	14.077	6.460	6.503	-0.359	1.048	-0.853	0.31	2.55

unique cost transfer allocation in the last step. Fairness is promoted by proposing the stable pre-nucleolus-based solution that minimizes the maximum player dissatisfaction.

Our results indicate that, for real-world instances, single-partition carpool arrangements offer simplicity and savings, which do not justify using complex multi-partitioned arrangements. This means that switching carpools for different trips does not offer such benefits as switching roles (between drivers and riders) within fixed carpools. Therefore, our recommended policy for real-world daily carpooling instances is to solve the 1-PART problem (a mixed-integer linear program) for as many restricted coalitions as possible, determine transfers among players (also by linear programs), and, if needed, supply the net transfer to the system to ensure stability.

Our article omitted some important issues that may constitute interesting directions for further research. For example, we assume that players only take into account driving costs (driving time, distance). However, other costs, such as riding times, may be important in the context of PDC and TFC carpooling. Including ride times is possible, but would require a slightly different setup, where each carpool member's ride times are recorded for each carpool and each route (the order in which different players' locations are visited) within the carpool. However, in the case of PDC, the issue of riding times can be heuristically solved as follows. Assume that in the optimal solution carpool $C = \{1, 2, 3, 4\}$ is formed with some non-zero share $\delta(C)$, and the shortest PDC path for this carpool (including the destination, denoted as 0) is 123401. Assume for simplicity that the driving distance is directly proportional to driving time. Due to symmetry of the metric, the total distance of this path is the same as the total distance of the following alternative path 432104, which is another shortest PDC path for the same carpool. However, these two paths mean different travel times for carpool participants. In particular, by alternating these two routes, we can ensure equal travel time for each participant. Indeed, the average riding time alternating these two routes is always equal to half the total driving time on each of them. This heuristic solution equalizes the travel time of carpooling participants.

Our model assumes that every commuter has a vehicle. We can relax this assumption by adding a constraint to the first stage problem (6) that the driving share of a person without a vehicle is 0 in every carpool that contains this person. Then, in the third step (15) we establish transfers (the price a commuter must pay to participate in the mechanism) that ensure incentive compatibility. The only restriction is that the total number of seats cannot be less than the number of commuters. The model assumes that the IR constraints are always met. Alternatively, they can be treated as soft restrictions (penalized in the objective) or restrictions that vary depending on the commuter's preferences. Such a modification would only change the Stage 1 problem.

Our approach focuses on a long time horizon in which optimal driving shares can be implemented. Changes to the mix of players and their locations require solving a new carpooling problem after each change. Solutions that require players not only to switch roles (rider and driver) in a given carpool, but also to switch carpools, may also cause difficulties and additional coordination costs. Although additional constraints can easily be added to the optimization problem, we show in our numerical simulations that the 1-PART IR solution, which does not require carpool switching, is very close to the M-PART IR solution. A cost-benefit analysis may show that in some cases the 1-PART IR solution is sufficient.

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Online Appendix

A Proofs and additional propositions

Proof of Proposition 1

In order to prove the result we must prove that the set of feasible solutions in 1-PART IR is a a subset of those in M-PART IR which in turn is a subset of those in NO-PART IR. We start with the latter inclusion. Note that the M-PART IR constraints (7), and (8) jointly imply the NO-PART IR constraint (19). To see it assume that the former hold. So, for each player i we can rewrite the latter as: $\sum_C \sum_P \theta(P)\chi'(P,C)\chi(C,i) = \sum_P \theta(P) \sum_C \chi'(P,C)\chi(C,i)$. Note that $\sum_C \chi'(P,C)\chi(C,i) = 1$ since P is a partition, so the previous expression reduces to $\sum_P \theta(P)$ which equals 1 by (8).

We now prove that the set of feasible solutions in 1-PART IR is a subset of those in M-PART IR. First note that the constraints (21), (22), and (23) in 1-PART IR imply that the collection of $C \in \mathcal{C}_m$ for which $\gamma(C) = 1$ forms a single partition of N: each individual belongs to exactly one carpool. Call this partition P^*. Thus $\theta(P^*) = 1$ which verifies (8) and by definition of χ', $0 = \delta(C) = \chi'(P^*, C) = 0$ for $C \notin P^*$, and $1 = \delta(C) = \chi'(P^*, C)$ for $C \in P^*$, which verifies (7).

Finally, we show that the bounds are tight. For TFC, Table 6 shows that there are instances in which the overall cost in M-PART IR is equal to NO-PART IR (lower bound) and/or to 1-PART IR (upper bound). The same is shown for PDC in Table 7.

Implications of a metric

Proposition 2. *In the TFC game, for any two locations* $1, s$ *and a non-empty set of other locations* $(2, ..., k)$*, the shortest path between* 1 *and* s *via all the locations in* $(2, ..., k)$ *is* $12...ks$ *if and only if the shortest path between* s *and* 1 *via all locations in* $(2, ..., k)$ *is* $sk...21$*.*

Proof. Directly from the symmetry of a metric. □

Proposition 3. *In the PDC game:*

a) *In each non-singleton carpool there are at least two members tied for driving.*

b) *The distance traveled is the same no matter whether the driver is located at its own location or at the destination point.*

c) *For any nonempty carpool C and any order of visiting its members, the distance of commuting together in this order is lower than the overall distance of each member of C driving separately.*

d) *Minimum cost subadditivity: For any nonempty carpools A, B, $d(A \cup B) \leq d(A) + d(B)$, where for $d(C)$ denotes the minimum distance PDC route for carpool $C \in N$, i.e. $d(C) = \min_{i \in C} c(C, i)$, where c is assumed to be a PDC cost.*

e) *Given two partitions P_1, P_2 of N. If P_1 is finer than P_2, then its overall minimum distance is not less than that of P_2.*

Proof. a) This is so because the distance of $1...is1$ is the same as that of $is1...i$, and by symmetry of a metric, the latter is the same as that of $i...1si$.

b) By symmetry, the distance of $1...is1$ is the same as that of $s1...is$.

c) Let $C = \{1, ..., k\}$ be a carpool. Suppose that the commuters in C drive together with a pre-specified (i.e., not necessarily the shortest) order of visited locations. Without loss of generality let the order be $12...ks1$. The distance corresponding to this route is the same as that of the route $s12...ks$. If all commuters in C drive separately, each of them follows the route isi, or equivalently sis. Hence, the overall route in this case can be written as $s1s2s...sks$. Thus, driving together saves distance because, as compared to each participant driving separately, it dispenses the need to visit many intermediate locations (s in between the commuters), which by triangle inequality will not make the distance smaller.

d) Let $C = \{1, ..., k\}$ be a carpool. Suppose that commuters in C drive together and player $f \in N \setminus C$ drives by itself. Let $\sigma(1)\sigma(2)...\sigma(k)s\sigma(1)$ be the path taken by C with distance

$d(C)$, where $\sigma : C \to C$ is the permutation of commuters in C which minimizes their distance. The distance traveled by f is the distance of route fsf, which by symmetry of a metric is the same as the distance of route sfs. Hence, the shortest total distance of carpools: C and f, i.e., $d(C) + d(f)$, is the distance traveled if the following route is taken: $s\sigma(1)\sigma(2)...\sigma(k)sfs$. On the other hand, we know by nonnegativity of a metric that $d(C \cup \{f\})$ is no greater than the distance of the following route: $s\sigma(1)\sigma(2)...\sigma(k)fs$. In fact, this is the optimal route for C with one additional location f visited just before s (which is one way of driving together in carpool $C \cup \{f\}$, albeit not necessarily the optimal one). The two paths differ in that the former has one extra intermediate location s, while the latter does not. Using the triangle inequality, we thus obtain that $d(C) + d(f) \geq d(C \cup \{f\})$. We can repeat this argument with adding more participants to C, finally proving the desired conclusion.

e) To prove it we use d). For two partitions P_1 and P_2 of N, if P_1 is finer than P_2 (i.e., for every $C \in P_1$, there is $C' \in P_2$ such that $C \subseteq C'$ and P_1 is distinct from P_2), then $\sum_{C \in P_1} v(C) \leq \sum_{C \in P_2} v(C)$. □

Note that point b) in Proposition 3 implies that the optimal distance in the PDC game is the same when a parent drives their children to school, picks up other children from the pool, drops off all children at the school, and returns home, or the school bus leaves the school, picks up all children from the pool, and returns to the school.

B Selecting pools and partitions for large instances

The number of set partitions increases rapidly with problem size. The following techniques allow finding good approximate solutions to large instances.

Number of m-restricted partitions of a set of n elements

To find the number of set partitions of n elements with a coalition size restricted to at most m elements, we use the following generating function (see, e.g., Flajolet and Sedgewick, 2009): $G(z) = \left(\exp\left(\frac{z^j}{j!}\right) - 1\right) \exp\left(\sum_{i=1}^{j-1} \frac{z^i}{i!}\right)$. Next, we find the coefficient of z^n in the expansion of G about the point $z = 0$ for each $j = 1, 2, .., m$ and multiply it by $n!$. Finally, we add these numbers together to obtain the m-restricted set partition count of n elements.

For $n = 12$ and $m = 4$, this results in $1 + 140151 + 1540440 + 1624425 = 3305017$. Note that number 1 represents a partition with all coalitions restricted to a single participant; it corresponds to the solution without carpooling, i.e., all participants drive alone.

Selecting candidate pools

The number of m-restricted non-empty coalitions that can be formed from a set of n participants ($m \leq n$) is a sum of binomial coefficients (see above). Although the total enumeration can be achieved even for instances with several hundred participants and a small pool size, the resulting linear programs are too large for most computers to solve.

The number of coalitions (carpools) for larger instances can be limited using the following pool-selection approach.

1. Enumerate all non-empty coalitions (pools) with at most m participants, and find the centroid of each pool (including single-participant "pools").

2. Weight each pool with the average distance from each participant's origin to the line passing through the destination and the pool's centroid.

3. If the angle between any two points of origin (with a vertex at the destination) of a pool is greater than $\pi/2$, add a penalty (a very large positive number) to the weight of that pool.

4. Add a penalty to all carpools, which contain points of origin located exactly at the destination.

5. Select a limited number of candidate pools no less than n and no more than, e.g., 300000 with the smallest weights. Note that all single "pools" have the weight equal to zero and will be included in the selection.

Selecting candidate pools $\hat{\mathcal{C}}_m$ allows us to find the lower bound (for this set of pools) on the objective of the carpooling game with individual rationality using a linear program, NO-PART IR (18). We also find the upper bound on the objective, i.e., a single-partitioned solution, with a mixed-integer program 1-PART IR (20).

Selecting candidate partitions

Before we begin the selection process, we apply *Algorithm X* on the set of coalitions found by NO-PART IR. If these coalitions form a full partition of the set of participants, this is the final solution, and the lower bound is the best-known objective, i.e., the minimum total distance driven by all participants for the selected pools.

Otherwise, following the solution algorithm, we consider partitions $\hat{\mathcal{P}}_m$ generated from a union of pools from the NO-PART IR, 1-PART IR and up to five alternative 1-PART IR solutions. The alternative 1-PART solutions are obtained with all previously-selected pools removed. Fewer than five alternative solutions are used if *Algorithm X* takes longer than, e.g., 5 hours to finish or generates more than a specified number (e.g. 10 million) partitions. Still, the smallest union of pools to be partitioned consists of the pools identified by the NO-PART IR and 1-PART IR in order to guarantee at least one full partition in the M-PART IR solution.

C Real-world instances

In this appendix we present the results of our computational experiments involving 35 *Daily Carpooling Problem* instances used in Baldacci et al. (2004) also known as the *University of Bologna Carpooling Instances.* Each of these instances consists of between 50 and 250 points of origin and an additional location designated as the destination. The original files are available from the University of Bologna http://astarte.csr.unibo.it/data/. Rather than using the distance matrices provided (rounded to integers), we computed accurate Euclidean distances for the original integer coordinates. We assumed that each point of origin has one commuter and a vehicle with $m = 4$ available seats. Instead of designating 25% of the commuters as drivers as suggested by Baldacci et al. (2004), we allowed any of the commuters to act as a driver. Also, we ignored the earliest leaving and latest arrival times, which led to instances A4 and A5 being identical.

Our M-PART IR model (6) yields an optimal solution to the carpooling game with individual rationality constraints, provided that all coalitions, as well as all set partitions, are given as inputs. While the total enumeration of carpools was possible for all of the real-world *Daily Carpooling Problem* instances (the number of all non-empty m-restricted carpools for $n = 50$ is 251175, and for $n = 250$, it is 161487125), this was not the case with set partitions. Even if the size of each coalition is restricted to at most 4, the number of set partitions grows extremely fast in the problem size, effectively preventing using all partitions for problems with more than 14 participants. Table 10 shows the number of pools with at most 4 participants, and the corresponding number of restricted set partitions (Bell B_n number with an m-restricted pool size) for the *Daily Carpooling Problem* instances determined by the formula presented in Appendix B.

Therefore, we limit both the number of coalitions and the number of partitions considered in the M-PART IR model using the approach described in Appendix B. However, with appropriate pool and partition selection methods, our heuristic approach results in significant savings from individually-rational carpooling.

Table 10: *Daily Carpooling Problem instances – pool and partition counts*

Instances	**n**	**Pools** $\lvert\mathcal{C}_4\rvert$	**Partitions** $\lvert\mathcal{P}_4\rvert$
A1	50	251175	4.03083×10^{45}
A2	75	1285825	6.74707×10^{76}
A3	100	4087975	3.68888×10^{110}
A4, A5	120	8502670	8.65512×10^{138}
A6	134	13241880	2.54159×10^{159}
A7	150	20822900	2.16735×10^{183}
A8, A9	170	34404345	9.36314×10^{213}
A10	195	59645495	1.83639×10^{253}
A11	199	64704850	4.41378×10^{259}
A12	225	105861225	4.98479×10^{301}
B1 - B7	100	4087975	3.68888×10^{110}
B8 - B12	150	20822900	2.16735×10^{183}
B12 - B18	200	66018450	1.75336×10^{261}
B19 - B23	250	161487125	1.00555×10^{343}

Tables 11–14 show the cost minimization results for the *Daily Carpooling Problem* instances for TFC and PDC carpooling models. A1 was the only instance for which we could use all pools to solve the NO-PART IR and 1-PART IR (for all other instances, the number of candidate pools is limited to 300000). For this reason, the reported LBs of 497.40 and 846.32 for TFC and PDC respectively, are the true (not heuristic) lower bounds on the carpooling with individual rationality problem and the corresponding 1-PART IR solution is the *optimal* single-partition solution to A1. However, the M-PART IR is a heuristic solution to A1, because it does not consider all 4.03083×10^{45} set partitions.

A1 pools generated by the NO-PART IR (see Figure 2) did not form a full partition of the set of participants, so we added pools from 1-PART IR and five alternative 1-PART IR solutions (see Figure 3), which resulted in $|\hat{\mathcal{C}}_4| = 138$ distinct pools. *Algorithm X* applied on the 1-PART IR and five alternative 1-PART IR solutions found $|\hat{P}_4| = 312$ and 78 candidate partitions for TFC and PDC respectively, which were used as input to M-PART IR.

The TFC M-PART IR model applied to A1 yielded a solution with four partitions ($p = 4$), and the objective value only 1.52% above the true lower bound. Such a carpooling arrangement results in 57.98% savings over the baseline, i.e., no carpooling.

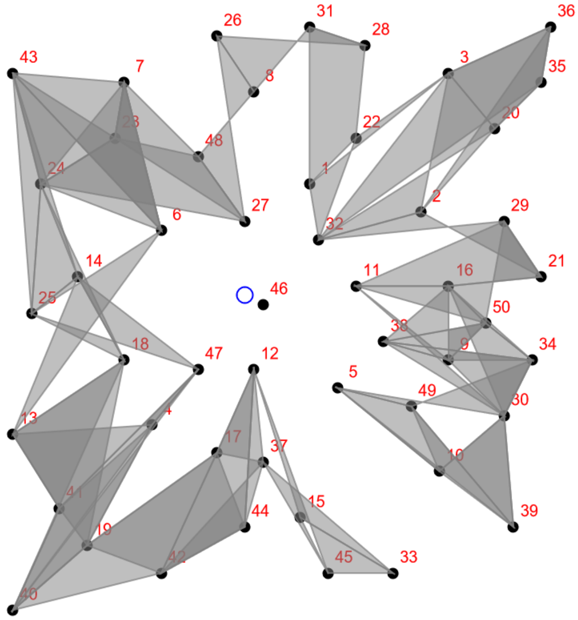

Figure 2: NO-PART IR Pools for A1 (TFC)

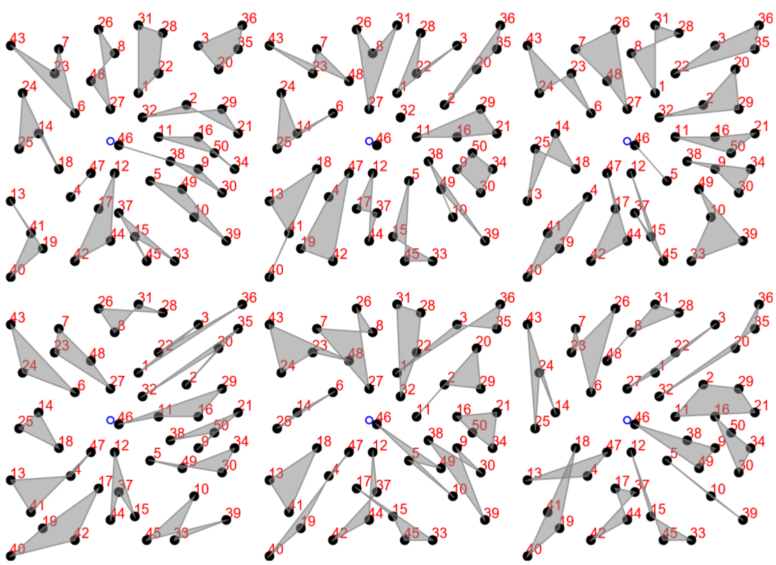

Figure 3: 1-PART IR and five alternative solutions for A1 (TFC)

Table 11: Individually rational (Step 1) solutions to *Daily Carpooling Problem* instances – TFC

Inst.	n	$\|\hat{\mathcal{C}}_4\|$	$\|\hat{\mathcal{P}}_4\|$	**Base**	**NO-PART IR (LB‡)**	**1-PART IR (UB‡)**	**M-PART IR**	p	**Sav. [%]**	**Gap [%]**	**M vs. 1 [pp]**
A1	50	138	312	1201.18	497.40	509.07	504.95	5	57.96	1.52	0.34
A2	75	194	556	1815.43	694.86	708.81	704.64	9	61.19	1.41	0.23
A3	100	266	2950	2494.71	904.52	920.30	914.72	8	63.33	1.13	0.22
A4*	120	306	3905664	6119.65	1726.35	1761.22	1760.12	8	71.24	1.96	0.02
A6	134	275	717408	4814.16	1660.56	1689.88†	1689.36	4	64.91	1.73	0.01
A7	150	389	229370	3680.25	1218.29	1235.47	1231.28	12	66.54	1.07	0.11
A8	170	352	91200	7783.70	2289.54	2312.95	2309.94	11	70.32	0.89	0.04
A9	170	384	2777088	6239.95	1887.70	1905.79	1903.15	14	69.50	0.82	0.04
A10	195	348	195456	8882.55	2612.52	2630.37	2627.37	14	70.42	0.57	0.03
A11	199	522	1743192	4804.21	1529.02	1543.60	1540.33	15	67.94	0.74	0.07
A12	225	523	81858	5507.45	1737.61	1755.20	1750.93	9	68.21	0.77	0.08
B1	100	232	506701	11308.05	3671.78	3901.77	3899.33	8	65.52	6.20	0.02
B2	100	254	338173	11659.45	4081.37	4144.90	4136.72	4	64.52	1.36	0.07
B3	100	231	274340	11795.95	4091.96	4172.01	4164.09	4	64.70	1.76	0.07
B4	100	230	448489	11883.00	3755.56	4004.04	4000.91	9	66.33	6.53	0.03
B5	100	231	149113	11735.95	3809.86	4052.86	4050.01	4	65.49	6.30	0.02
B6	100	238	44891	11756.45	4139.05	4174.10	4169.35	5	64.54	0.73	0.04
B7	100	232	98180	11880.20	4105.83	4148.74	4143.30	6	65.12	0.91	0.05
B8	150	263	440642	18711.60	5760.48	5861.42	5857.64	10	68.70	1.69	0.02
B9	150	265	76052	18630.60	6312.67	6459.80	6456.80	4	65.34	2.28	0.02
B10	150	309	2376385	18804.60	5657.15	5867.80	5864.54	6	68.81	3.67	0.02
B11	150	351	3136848	18604.80	6100.23	6410.00	6405.55	6	65.57	5.01	0.02
B12	150	305	1397922	18621.05	5972.70	6256.07	6248.74	7	66.44	4.62	0.04
B13	200	331	48000	25636.60	8443.45	8970.10	8969.57	6	65.01	6.23	0.00
B14	200	315	2161	25285.10	8898.65	†	9181.30	3	26.71	0.80	0.00
B15	200	308	2881	25591.35	9276.55	9424.07	9422.51	7	63.18	1.57	0.01
B16	200	360	243540	25487.70	8548.12	8627.06	8626.06	5	66.16	0.91	0.00
B17	200	354	77760	25699.60	8628.20	8771.26†	8769.48	6	65.88	1.64	0.01
B18	200	418	5167693	25596.95	8807.34	9090.47	9088.70	9	64.49	3.19	0.01
B19	250	448	145280	34162.55	10721.34	10793.74	10792.45	10	68.41	0.66	0.00
B20	250	445	4769281	34562.15	11006.32	11289.78	11287.86	8	67.34	2.56	0.01
B21	250	375	73731	34235.80	11507.69	†	11606.97	5	66.10	0.86	0.00
B22	250	442	290160	34314.35	12089.22†	†	12233.71	9	64.35	1.20	0.00
B23	250	453	251136	34348.95	10894.32	11020.33†	11018.56	5	67.92	1.14	0.01

* A5 is identical to A4
† Optimization stopped after 5 hours
‡ For the selected subset of 300000 pools

For the TFC model the savings from Step 1 for the *Daily Carpooling Problem* instances ranged from 57.96% to 71.24%, with an average of 66.19%. 1-PART IR was stopped after 5 hours for 4 out of 35 instances. In three cases (B14, B21 and B22), no feasible 1-PART solution was obtained within the time limit. Still, even for these instances, the M-PART IR yielded the best-known solutions. The average time to solve a single NO-PART IR problem was 165.46 seconds and the average time to solve the M-PART IR was 42.37 seconds for TFC carpooling.

For the PDC model, the savings from Step 1 for the *Daily Carpooling Problem* instances ranged from 63.65% to 72.31%, with an average of 67.65%. 1-PART IR was stopped after 5 hours for 10 out of 35 instances. In just one case (B11), no feasible 1-PART solution was obtained within the time limit. Still, the M-PART IR yielded the best-known solutions for all instances. The average time to solve a single NO-PART IR problem was 170.18 seconds and the average time to solve the M-PART IR was 45.92 seconds for PDC carpooling.

Tables 11 and 12 show that substantial savings over the baseline can be achieved even with a partial enumeration of carpools and set partitions. In fact, the average savings (66.18% for TFC and 67.63% for PDC) were substantially higher than those for the simulated smaller instances (41.89% for TFC and 54% for PDC). This suggests that the gain in savings due to increasing the size of the problem is so large that it offsets by a large margin the loss of using only a partial enumeration due to increased complexity. Tables 13 and 14 show that the cost of IC is generally larger for real-world instances than for the simulated instances. It suggests that meeting the IC conditions is increasingly costly in the size of the problem.

Table 12: Individually rational (Step 1) solutions to *Daily Carpooling Problem* instances – PDC

Inst.	n	$\lvert\hat{\mathcal{C}}_4\rvert$	$\lvert\hat{\mathcal{P}}_4\rvert$	**Base**	**NO-PART IR (LB‡)**	**1-PART IR (UB‡)**	**M-PART IR**	p	**Sav. [%]**	**Gap [%]**	**M vs. 1 [pp]**
A1	50	129	78	2402.35	846.32	858.35	858.13	2	64.28	1.40	0.01
A2	75	194	232	3630.86	1226.63	1234.30	1234.05	2	66.01	0.60	0.01
A3	100	253	588	4989.42	1618.79	1626.50	1626.50	1	67.40	0.48	0.00
A4*	120	296	3674352	12239.30	3310.19	3388.58†	3388.45	2	72.31	2.36	0.00
A6	134	275	2352960	9628.31	3229.05	3285.72†	3285.70	2	65.87	1.75	0.00
A7	150	387	61478	7360.50	2253.54	2264.62	2264.62	1	69.23	0.49	0.00
A8	170	399	2586144	15567.40	4415.87	4445.38	4444.96	2	71.45	0.66	0.00
A9	170	391	1003464	12479.90	3548.24	3581.24	3581.24	1	71.30	0.93	0.00
A10	195	444	4779112	17765.10	5037.05	5063.50	5063.50	1	71.50	0.53	0.00
A11	199	506	766768	9608.41	2862.38	2875.66	2875.66	1	70.07	0.46	0.00
A12	225	510	74856	11014.90	3246.19	3261.58	3261.58	1	70.39	0.47	0.00
B1	100	228	1314577	22616.10	6994.44	7492.03	7491.28	3	66.88	7.10	0.00
B2	100	258	139313	23318.90	7806.18	7909.51	7908.80	2	66.08	1.31	0.00
B3	100	229	47320	23591.90	7876.52	7957.28	7957.07	2	66.27	1.02	0.00
B4	100	223	228853	23766.00	7178.21	7632.08	7631.93	2	67.89	6.32	0.00
B5	100	230	18106	23471.90	7294.32	7766.73	7766.73	1	66.91	6.48	0.00
B6	100	227	84850	23512.90	7826.81	7877.25	7872.84	3	66.52	0.59	0.02
B7	100	234	80480	23760.40	7840.10	7915.83	7915.61	3	66.69	0.96	0.00
B8	150	301	3619442	37423.20	11204.81	11400.20†	11400.20	1	69.54	1.74	0.00
B9	150	301	62762	37261.20	12298.72	12574.69†	12574.61	4	66.25	2.24	0.00
B10	150	345	3171361	37609.20	10970.89	11411.12†	11411.12	1	69.66	4.01	0.00
B11	150	325	52164	37209.60	11937.25	†	12611.02	5	66.11	5.64	0.00
B12	150	262	353810	37242.10	11605.33	12138.96	12138.96	3	67.41	4.60	0.00
B13	200	412	6128118	51273.20	16574.31	17590.97	17590.97	1	65.69	6.13	0.00
B14	200	362	2773	50570.20	17551.27	18068.65	18068.65	1	26.86	0.80	0.00
B15	200	313	4321	51182.70	18278.36	18606.79†	18606.69	8	63.65	1.80	0.00
B16	200	357	291924	50975.40	16812.34	16972.82	16972.82	2	66.70	0.95	0.00
B17	200	411	83052	51399.20	16936.04	17209.24†	17209.24	2	66.52	1.61	0.00
B18	200	358	97441	51193.90	17251.14	17847.02	17847.02	2	65.14	3.45	0.00
B19	250	513	3286464	68325.10	21112.13	21270.15	21270.11	2	68.87	0.75	0.00
B20	250	444	3234817	69124.30	21679.84	22243.52	22243.52	2	67.82	2.60	0.00
B21	250	442	1849459	68471.60	22642.55	22851.25	22850.49	6	66.63	0.92	0.00
B22	250	448	55808	68628.70	23901.93	24187.76†	24187.77	3	64.76	1.20	0.00
B23	250	506	1266216	68697.90	21400.66	21622.08†	21622.08	3	68.53	1.03	0.00

* A5 is identical to A4

† Optimization stopped after 5 hours

‡ For the selected subset of 300000 pools

Table 13: The stable pre-nucleolus (Step 3) solution for *Daily Carpooling Problem* instances – TFC

Inst.	**Base**	**M-PART IR**	**Net Tr.**	**Max Tr.**	**Min Tr.**	**Cost of IC [pp.]**
A1	1201.18	504.952	-126.854	18.718	-23.520	10.56
A2	1815.43	704.642	-166.420	20.000	-27.005	9.17
A3	2494.71	914.725	-172.685	16.852	-25.896	6.92
A4*	6119.65	1760.116	-468.283	37.357	-68.610	7.65
A6	4814.16	1689.360	-640.570	78.910	-81.318	13.31
A7	3680.25	1231.276	-376.300	19.843	-30.055	10.22
A8	7783.70	2309.941	-467.865	54.630	-68.610	6.01
A9	6239.95	1903.153	-736.436	38.134	-57.741	11.80
A10	8882.55	2627.369	-700.817	53.740	-68.610	7.89
A11	4804.21	1540.325	-523.240	25.495	-30.055	10.89
A12	5507.45	1750.926	-304.014	22.668	-30.055	5.52
B1	11308.05	3899.328	-1071.450	440.624	-332.969	9.48
B2	11659.45	4136.720	-767.942	253.466	-453.987	6.59
B3	11795.95	4164.085	-1985.630	231.877	-387.137	16.83
B4	11883.00	4000.910	-1595.440	273.762	-412.138	13.43
B5	11735.95	4050.013	-2368.780	289.675	-1091.560	20.18
B6	11756.45	4169.354	-2325.170	235.194	-358.943	19.78
B7	11880.20	4143.296	-990.893	188.085	-363.119	8.34
B8	18711.60	5857.638	-472.831	322.025	-372.538	2.53
B9	18630.60	6456.802	-194.527	423.524	-423.574	1.04
B10	18804.60	5864.544	-1219.180	319.708	-454.779	6.48
B11	18604.80	6405.551	-3057.100	343.607	-379.674	16.43
B12	18621.05	6248.739	-1160.830	644.881	-412.925	6.23
B13	25636.60	8969.573	-2021.960	286.339	-361.542	7.89
B14	25285.10	9181.302	-846.967	542.987	-364.253	3.35
B15	25591.35	9422.506	-647.061	591.238	-247.979	2.53
B16	25487.70	8626.057	-1019.540	331.976	-279.783	4.00
B17	25699.60	8769.484	-2123.240	426.368	-431.898	8.26
B18	25596.95	9088.697	-245.154	538.666	-390.194	0.96
B19	34162.55	10792.450	-2556.640	237.504	-453.935	7.48
B20	34562.15	11287.855	-1147.840	357.069	-441.714	3.32
B21	34235.80	11606.966	0.000	597.509	-349.453	0.00
B22	34314.35	12233.712	-3884.330	377.397	-262.532	11.32
B23	34348.95	11018.562	-2413.990	322.803	-435.313	7.03

* A5 is identical to A4.

Table 14: The stable pre-nucleolus (Step 3) solution for *Daily Carpooling Problem* instances – PDC

Inst.	**Base**	**M-PART IR**	**Net Tr.**	**Max Tr.**	**Min Tr.**	**Cost of IC [pp.]**
A1	2402.35	858.133	-166.791	36.251	-51.613	6.94
A2	3630.86	1234.050	-331.323	46.615	-57.505	9.13
A3	4989.42	1626.501	-488.696	42.332	-57.528	9.79
A4*	12239.30	3388.451	-1300.510	97.755	-139.272	10.63
A6	9628.31	3285.700	-1854.080	129.799	-163.739	19.26
A7	7360.50	2264.623	-977.355	38.287	-61.573	13.28
A8	15567.40	4444.963	-1343.970	159.620	-139.272	8.63
A9	12479.90	3581.241	-1837.900	112.641	-117.386	14.73
A10	17765.10	5063.503	-1928.860	105.906	-139.272	10.86
A11	9608.41	2875.663	-1348.700	53.022	-61.573	14.04
A12	11014.90	3261.585	-1313.600	57.737	-62.632	11.93
B1	22616.10	7491.278	-1690.160	1100.870	-750.255	7.47
B2	23318.90	7908.800	-779.697	563.489	-752.117	3.34
B3	23591.90	7957.069	-5506.130	653.903	-779.623	23.34
B4	23766.00	7631.925	-961.622	712.306	-829.978	4.05
B5	23471.90	7766.735	-2408.420	635.959	-557.650	10.26
B6	23512.90	7872.838	-4897.120	931.838	-729.716	20.83
B7	23760.40	7915.608	-3494.580	546.619	-741.637	14.71
B8	37423.20	11400.198	-1993.060	623.982	-772.990	5.33
B9	37261.20	12574.612	0.000	889.709	-853.402	0.00
B10	37609.20	11411.116	-2905.220	639.415	-917.367	7.72
B11	37209.60	12611.023	-5981.470	730.069	-761.778	16.08
B12	37242.10	12138.956	-1079.370	1146.910	-829.115	2.90
B13	51273.20	17590.967	-6859.130	764.019	-727.496	13.38
B14	50570.20	18068.651	-1379.510	793.204	-641.568	2.73
B15	51182.70	18606.690	-712.068	1072.210	-506.217	1.39
B16	50975.40	16972.815	-5092.020	727.964	-585.752	9.99
B17	51399.20	17209.241	-3573.050	929.048	-770.687	6.95
B18	51193.90	17847.018	-1981.140	895.842	-781.998	3.87
B19	68325.10	21270.105	-11380.600	698.584	-910.725	16.66
B20	69124.30	22243.518	-2181.590	1138.450	-731.666	3.16
B21	68471.60	22850.490	-1697.570	1301.080	-627.991	2.48
B22	68628.70	24187.765	-7873.490	755.695	-534.704	11.47
B23	68697.90	21622.082	-6618.680	650.890	-870.830	9.63

* A5 is identical to A4.