Discounted Incremental Utility

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Abstract

Most decisions involve multiple payoffs over time and under risk, as for instance the sequential play of a lottery. For an individual who cares not only about profit at the end, but also on how early these profits accrue, we apply a modified version of the axioms of subjective expected utility to obtain the Discounted Incremental Utility model. To this classical bedrock, we add the notion that preferences are affected by range effects. This modification results in the so-called Range-Discounted Incremental Utility, which can successfully predict a plethora of phenomena such as attitudes towards sequential play, the four-fold pattern, preference for temporal hedging, preference reversals for risk and time, and patterns of decreasing impatience. For comparison, we also introduce a rank-dependent modification and examine its shortcomings.

Keywords: Risk and Time Preferences, Samuelson Paradox, Range Effects, Decreasing Impatience, Rank-dependent Utility.

JEL codes: D91, D81, C91

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1 Introduction

The most economically significant problems are related to uncertainty and time delay. The most popular model used to evaluate uncertain income streams is the discounted expected utility (DEU, in short, see Fishburn, 1970, Theorem 11.4), which combines the idea of exponential time discounting (Koopmans, 1960) with the expected utility of risky lotteries (Von Neumann and Morgenstern, 1945; Savage, 1954). Let $(X_0, X_1, ..., X_T)$ denote a risky payoff stream where X_t denotes a random payoff received at time t and p_t its distribution. DEU preferences are represented by the following functional:

$$(X_0, X_1, ..., X_T) \mapsto \sum_{t=0}^{T} \delta^t \mathbb{E}_{p_t} u(X_t),$$

where $\delta \in (0,1)$ is a discount factor, u is the utility function, and $\mathbb{E}_{p_t}$ denotes expectation with respect to the distribution of X_t.

This model implies an extreme form of myopia. If the decision maker dislikes an equal chance of winning \$300 dollars or losing \$200 (let's call it the p bet), then under DEU he should dislike it even more if this bet is sequentially repeated over several periods irrespective of the correlation between consecutive draws.

the following risky payoff stream even more:

\$300 today, −\$200 tomorrow	with probability 0.5
−\$200 today, \$300 tomorrow	with probability 0.5.

We can think of this stream as accepting p for just one day. After a day, the bet is canceled and winnings returned and you additionally get \$100 for sure. The stream is clearly better than p.

Sequential vs. simultaneous play Consider an equal chance prospect of winning \$300 or losing \$200. In a famous example, Samuelson (1963) reports his colleague who rejects a single play of this prospect, but is willing to accept several repetitions. If the repetitions and the payout occur simultaneously, these preferences can be accommodated by DEU. But what if these repetitions are spread over time with money being paid out after each repetition. DEU predicts that if a single play is rejected, then the sequential ones are always rejected. In fact they are even worse than a single play. This is so because DEU evaluates each period independently of another so if a lottery played in one period is unattractive, its repetition in another period must also be so. Benartzi and Thaler (1995) echo this line of thought, which they coin myopic loss aversion.

We tested this prediction on a group of 140 MBA students from the University of Virginia and found that even though 64% of subjects rejected a single play, only 40% rejected 10 sequential repetitions spread over 10 weeks starting from now. Thus, a majority of subjects do not exhibit myopic loss aversion, and *at least* 24% of subjects rejected a single play **and** accepted a sequential play, thus violating the DEU prediction.

Correlation neutrality The sequential play above assumed iid draws. How about serially correlated repetitions? DEU predicts correlation neutrality (Fishburn, 1970, Theorem 11.1), i.e. if marginal payoff distributions in each period are the same in two streams, then these streams are preferentially equivalent. Lanier et al. (in press) as well as Rohde and Yu (2022) reject this prediction and find that people have preference for negative correlation, or – stated differently – they like to hedge against inter-temporal risk. For example, people exhibit strong preference for an equal chance of getting

$$(\$300 \text{ today}, -\$200 \text{ in a week}) \text{ or } (-\$200 \text{ today}, \$300 \text{ in a week})$$

as compared to an equal chance of:

$$(\$300 \text{ today}, \$300 \text{ in a week}) \text{ or } (-\$200 \text{ today}, -\$200 \text{ in a week}).$$

Discounted Incremental Utility While descriptively problematic, correlation neutrality in DEU is analytically convenient. It allows treating each period separately by taking expectation with respect to the marginal at time t instead of the joint distribution over payoff streams. Our aim is to propose a model that shares this analytical convenience with DEU while predicting things like preferences for sequential play or correlation aversion. One such model, commonly adopted in the finance literature (Vila and Zariphopoulou, 1997), is to take the expected utility of the sum of payoffs at some horizon. It clearly captures the phenomena discussed above. However, since it cares only about the aggregate at the horizon, it ignores the time dimension and is completely insensitive to period risk. The model we propose combines the nice features of the EU-sum approach with the analytical convenience of DEU.

Instead of the distribution of incremental payoffs we propose to focus on the balance of payoffs at a given time period, i.e. the distribution of the accrued payoffs from the beginning of the stream till current period. We do not ignore the time dimension because we consider the distribution of the accrued payoffs period by period. In this way we may still look at the distribution in a given time period instead of the distribution of the whole payoff stream. Let

The Discounted Incremental Utility model evaluates

$$b \mapsto \sum_{t \in \mathcal{T}} \delta(t) \mathbb{E}_{b_t} \left[u\left(\sum_{i=0}^{t} x_i \right) - u\left(\sum_{i=0}^{t-1} x_i \right) \right].$$

One way, called the payoff vector approach (Kreps and Porteus, 1978, p.185), is to define a choice object as a probability distribution over all payoff paths in a risky payoff stream $\mathbf{X}$.[1] Such domain is appropriate to model pure time preference (**when** the money is paid). The payoff vector approach, however, fails to capture the fear of termination (**whether** the money is paid).

because it implies two kinds of separability: additive separability in time and multiplicative separability with respect to time discounting. These properties lead to the predictions discussed above that are too restrictive to match the data we have. The model we propose, Discounted Incremental Utility, is a telescopic version of EU-sum, whereby period payoffs do matter individually, as well as in the aggregate. It exhibits a more nuanced form of myopic loss aversion, as well as a preference for negative serial correlation. Our key idea is to have separability on the level of cumulative rather than incremental payoffs. We now explain how and why this idea works.

The common feature in models with time delay is that of impatience. There are two main approaches to impatience. The first one, known at least since Samuelson (1937), hereinafter referred to as *pure time preference*, focuses on **when** a good is received. The impatient decision-maker prefers to receive the goods sooner rather than later. Under this approach it is standard to have preferences defined on the set of stochastic processes with values interpreted as *incremental payoffs* received at any time period. DEU is a leading model falling under this approach.

The second approach introduced by Halevy (2008) and followed by Baucells and Heukamp (2012) and Blavatskyy (2016), hereinafter called *fear of termination*, focuses on **whether** a good is received. According to this view people are impatient because although the present is certain, the future is inherently uncertain, and the further away in time, the less certain is the chance of getting the payout (hazard of mortality, a promise of a future reward may be breached, etc.). This approach focuses on *cumulative payoffs* received up to a termination time. The key role of cumulative instead of incremental payoffs makes this approach a good candidate to relax the DEU separability properties. Yet, the existing papers falling under the fear of termination approach assume a domain that is too narrow to address sequential repetitions of a risky prospect with or without serial correlation. Specifically, Halevy (2008) and Blavatskyy (2016) focus on *positive* payoffs under certainty (except for an objective risk of termination) while Baucells and Heukamp (2012) considers the restricted domain of delayed binary lotteries with one nonzero outcome.

We thus extend the fear of termination approach to both positive and negative payoffs, and arbitrary risks. Our domain of risky payoff streams is very broad and has one-time lotteries, payoff streams under certainty, and lotteries repeated over time—possibly with serial correlation—as special cases. In order to apply the fear of termination approach, we associate each risky payoff stream with a new object called a temporal profit profile, described by the probability distribution of accrued profits up to each time period. Each of these distributions is characterized by ob-

[1] Or a corresponding distributions under the recursive formulation of Kreps and Porteus (1978) and Epstein and Zin (1989), suitable for dynamic settings.

jectively given probabilities. To this objective probabilities, the individual overlaps a *subjective* probability—expressing the temporal fear of termination—that drives impatience. Thus, the reason he may prefer sooner positive payoffs is a higher guarantee of survival at such early time. In contrast, Halevy (2008) takes such fear as an objectively given mortality risk.

Taking profits (accumulated payoffs) instead of individual payoffs as the consequence is eminently realistic for finance and business decision, where accounting systems are geared towards aggregating payoffs and presenting profits. Borrowing from business language, cumulative payoffs up to any given time t will be called profits at time t; and they will be in the black if positive, and in the red if negative.

Our domain of probability distributions over the accrued profits offers a convenient setup for building preference foundations for the joint risk and time domain. In particular, observe that temporal profit profiles technically coincide with Anscombe-Aummann acts (Anscombe and Aumann, 1963). Thus, our first contribution is to adapt the Anscombe-Aummann axioms (Gilboa, 2009, p.143) of subjective expected utility to the temporal setting and obtain preference foundations for the *Discounted Incremental Utility* model,or DIU for short.[2] DIU involves a utility function for outcomes and a discount function for time. The individual discounts the expected incremental utility of each period, i.e., the utility of the profit up to today minus the utility of the previous profit. The model particularizes into simpler forms, including a novel discounted Kelly criterion, or EU-sum for individuals without any fear of termination.

On the descriptive side, DIU accommodates preference for sequential play and for negative correlation. For one-time lotteries, DIU, DEU, and EU-sum all coincide; and for a single lottery played today, DIU reduces to EU.

We focus on monetary consequences, relevant for many financial and business decisions, as well as for interpreting the results of laboratory experiments that commonly use monetary payoff. The formulation, however, could be extended to consumption.[3]

[2]DIU was hinted by Bell (1974, Eq. 4) based not on axiomatic properties, but on the requirement that if two payoffs are received in nearby times, then their utility should be similar to that of receiving their sum at once. Blavatskyy (2016) also argues for discounted incremental utility, but his setup is restricted to positive cash flows under certainty.

[3] A simple approach is to assume a quasi-linear utility model for consumption under certainty, so that the period's payoff is money plus the utils from consumption received under conditions of certainty (see Halevy (2008)). Such consumption utility should account for the desire to smooth out consumption. DIU then introduces a utility for risk and a discount function to account for risk and time preferences in an integrated way, as time is seen as inherently uncertain.

2 Discounted Incremental Utility for Risky Payoff Streams

2.1 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A risky payoff X is a simple random variable, that is real-valued function on Ω that has finite range and is measurable $\mathcal{F}$.[4]. The law of X, is a function $p(X) : \mathbb{R} \to [0,1]$ such that $p(X)(z) = \mathbb{P} \circ X^{-1}(z)$ for all $z \in \mathbb{R}$. Clearly, $\sum_{x \in A} p_X(x) = 1$ for a finite set A because X is simple. Let Δ denote the set of such probability distributions.

Our focus is not on the temporal resolution of uncertainty (see Kreps and Porteus, 1978), hence we assume the resolution time agrees with the reception time. We let $\mathcal{I} = \{0, 1, ..., T\}$ be the index set of times at which payoffs are received. These do not need to be equally spaced in the time line. From now on, the time frame remains fixed.

A *risky payoff stream* $\mathbf{X}$ is a collection of risky payoffs, $\mathbf{X} = (X_t : \Omega \to \mathbb{R})_{t \in \mathcal{I}}$, adapted to some filtration[5] $(\mathcal{F}_t, t \in \mathcal{I})$ of $\mathcal{F}$, i.e. X_t is measurable $\mathcal{F}_t$ for each $t \in \mathcal{I}$. Let $\mathcal{X}$ be the set of all risky payoff streams, and $\mathbb{E}$ denote the expectation wrt $\mathbb{P}$.

The values of a risky payoff stream designate incremental payoffs received in each time period. Associated with a risky payoff stream $\mathbf{X} = (X_0, X_1, ..., X_T)$ is its balance $b(\mathbf{X})$ defined as

$$b(\mathbf{X}) = \left(p(X_0), p(X_0 + X_1), ..., p\left(\sum_{i=0}^{T} X_i \right) \right).$$

The set of all balances is $\Delta^{\mathcal{I}}$. The balance traces the past, instead of focusing solely on the present, ignoring dependence on the state: for each period it determines the distribution of profit earned up to that period. Given two risky payoff streams $\mathbf{X}, \mathbf{Y}$ and a scalar α in $[0,1]$ we define $(\mathbf{X}, \mathbf{Y}; \alpha)$ as some risky payoff stream such that $b(\mathbf{X}, \mathbf{Y}; \alpha) = \alpha b(\mathbf{X}) + (1 - \alpha) b(\mathbf{Y})$.

Figure 1 shows an exemplary risky payoff stream depicted as a tree. The black numbers are incremental payoffs received at a given time period. The blue numbers indicate the accrued profits over time. Table 1 shows the associated balance.

2.2 A key modeling assumption

Let $\succcurlyeq$ be a preference over risky payoff streams. Our background assumption states that the decision maker is indifferent between two risky payoff streams if their balances are equal. Formally,

A0 $b(\mathbf{X}) = b(\mathbf{Y})$ implies $\mathbf{X} \sim \mathbf{Y}$.

[4] I.e. $[\omega \in \Omega : X(\omega) = x] \in \mathcal{F}$ for each real x.

[5] A filtration of $\mathcal{F}$ is a sequence $(\mathcal{F}_t, t \in \mathcal{I})$ of σ-fields satisfying $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_T \subseteq \mathcal{F}$.

Figure 1: A risky payoff stream. Accrued profits are shown in blue.

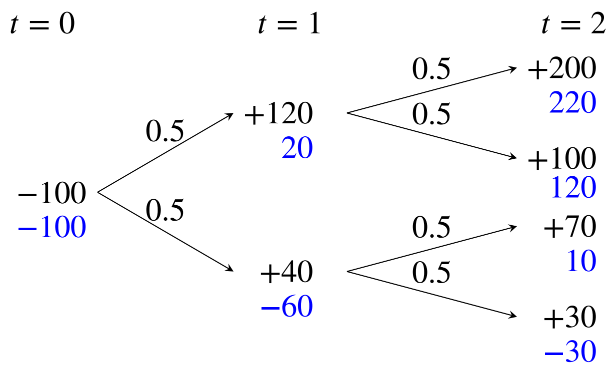


Table 1: Balance Associated with the Risky Payoff Stream in Figure 1

x	a_0	a_1	a_2
-100	100%		
-60		50%	
-30			25%
10			25%
20		50%	
120			25%
220			25%

A decision maker whose preferences over risky payoff streams satisfy **A0** only cares about the current balance associated with the stream. What matters is not the profit or loss in a given period, but being in the red or in the black, counting from the beginning of the stream. Moreover, it is only probability distribution of the accrued payoffs that matters rather than specific states or paths leading to these payoffs.

2.3 Fear of termination and the balance

Someone who satisfies assumption **A0** is interested in the balance of a given risky payoff stream in each period t, that is, the probability distribution of cumulative payoffs accrued from time 0 to t. Such a person differs from a person who is interested in the distribution of payoffs received only at time t by the reference point against which profits and losses are defined. In the first case, the reference point for the decision-maker's wealth at any time t is constant and equal to the wealth at time 0, while in the second case it is always adapted to the level of wealth from the previous period $t - 1$.

Suppose the decision maker facing a risky payoff stream $\mathbf{X} = (X_0, X_1, ..., X_T)$ cares only how much money he will get in total and not when this money is received. Then what matters to him is the sum of risky payoffs $\sum_{i=0}^{T} X_i$. However, payoffs that occur later in time are less certain that those that occur earlier. To account for that we may consider the possibility that the stream terminates (for whatever reason) before its natural horizon at $t < T$. In this case, he will be paid only up to this time. $\sum_{i=0}^{t} X_i$

We follow the common approach of assuming independence of the states in Ω (i.e., only probability distribution matters, not the particular states that generate those distributions).

Consider the following extension: given a risky payoff stream $\mathbf{X} = (X_0, ..., X_T)$, let

$$((X_0, ..., X_t, 0, ..., 0), t \in \mathcal{I})$$

be a collection of truncated streams, each reflecting the possibility that payoffs may be terminated at any time $t \in \mathcal{I}$. If the decision maker believes termination will surely occur at time T, only the non-truncated stream $(X_0, ..., X_T)$ matters. If, on the other hand, the decision maker exhibits fear of termination, then what matters conditional on termination occurring at time t is not when payoffs occur, but the profit accrued up to time t, i.e. $\sum_{i=0}^{t} X_i$. Accordingly, our object of choice is $a(\mathbf{X})$, which for each termination state t gives a probability distribution of accrued profits at t.

The example in Figure 1 is useful to illustrate what A0 entails: The decision maker would not mind if in $t = 2$ he instead receives an equal chance between -10 and -50 following the payoff of 120; and an equal chance between 180 and 280 following the 40, because the three probability distributions in Table 1 remain the same.

Axiom A0 implies two things. First, pure time preference and fear of termination are exchangeable explanations of the same preferences; a given level of impatience may be attributed either to time discounting or to holding subjective fear of termination. While these two motives could coexist, we can only identify them jointly. Second, independence of Ω is imposed on the level of accrued profits. This allows to adapt the Anscombe-Aumann two-stage uncertainty to our setup.

With A0 in mind we will describe the rest of axioms in terms of temporal profit profiles. Then state the main representation for $\succcurlyeq$, followed by the associated representation for $\succcurlyeq'$. Depending on the context, we refer to $t \in \mathcal{I}$ as a time period or a termination state interchangeably.

Specifically, we may construct a preference relation $\succcurlyeq^*$ over $\Delta^{\mathcal{I}}$ by writing $a \succcurlyeq^* b$ if and only if there exist $\mathbf{X}$ and $\mathbf{Y}$ in $\mathcal{X}$ such that $b(\mathbf{X}) = a$ and $b(\mathbf{Y}) = b$, and $\mathbf{X} \succcurlyeq \mathbf{Y}$. $\succcurlyeq^*$ is well-defined and inherits many properties from $\succcurlyeq$. First, $\succcurlyeq^*$ does not depend on the particular choice of risky payoff streams used to construct it. Suppose that $b(\mathbf{X}) = b(\mathbf{X}') = a$. Then, by **A0**, $\mathbf{X} \sim \mathbf{X}'$ and, since $\succcurlyeq$ is transitive, $\mathbf{X} \succcurlyeq \mathbf{Y} \iff \mathbf{X}' \succcurlyeq \mathbf{Y}$. So $\succcurlyeq^*$ will never run into contradictions. Second, for any $a \in \Delta^{\mathcal{I}}$ there is $\mathbf{X} \in \mathcal{X}$ such that $b(\mathbf{X}) = a$. Thus, since $\succcurlyeq$ is complete, so is $\succcurlyeq^*$. The same is true for transitivity.

2.4 Preferences and Representation

To simplify notation we embed $\mathbb{R}$ and Δ in $\mathcal{A}$, i.e. a consequence $x \in \mathbb{R}$ is identified with the profile in which x is paid in each period, and a risky prospect p is identified with the profile in which p is played in each period. Thus, the outcome '0' refers getting 0 in all periods. We also write (p, t) to denote the profile associated with playing a lottery $p \in \Delta$ at time t, and getting 0 in all other time periods. The mixture of profiles, $\alpha a + (1 - \alpha)b \in \mathcal{A}$, $\alpha \in [0, 1]$, is defined in the standard point-wise way.

A1 **Weak order:** $\succcurlyeq$ is complete and transitive.

A2 **Independence:** $\forall a, b, c \in \mathcal{A}$, if $a \succ b$ and $\alpha \in (0, 1]$, then $\alpha a + (1 - \alpha)c \succ \alpha b + (1 - \alpha)c$.

A3 **Archimedean:** $\forall a, b, c \in \mathcal{A}$, if $a \succ b$ and $b \succ c$, there $\exists \alpha, \beta \in (0, 1)$ such that $\alpha a + (1 - \alpha)c \succ b$ and $b \succ \beta a + (1 - \beta)c$.

To introduce next axioms, we define a new binary relation on $\Delta(C)$ which is also denoted by $\succcurlyeq$. For $p, q \in \Delta(C)$ we write $p \succcurlyeq q$ if and only if $a \succcurlyeq b$ for $a, b \in \mathcal{A}$ such that $a(t, \cdot) = p$ and $b(t, \cdot) = q$ for all $t \in \mathcal{T}$. Clearly, if $\succcurlyeq$ on $\mathcal{A}$ satisfies Ai, so does $\succcurlyeq$ on $\Delta(C)$ for $i = 1, 2, 3$.

A4 **Outcome monotonicity:** $\forall x, y \in \mathbb{R}$, if $x > y$, then $\delta_x \succ \delta_y$.

A5 **Temporal monotonicity:** $\forall p, q \in \Delta(C)$ and $\forall t \in \mathcal{T}$, if $(p, t) \succ (q, t)$ then $p \succ q$.

A6 **Impatience:** For all $x \in C$ and $t, t' \in \mathcal{T}$, $t < t'$, if $x > 0$ then $(\delta_x, t') \succcurlyeq (\delta_x, t)$.

This seven axioms combine the normative vNM axioms typically invoked in the context of risk with the standard temporal axioms of impatience and risk and time independence. The novelty lies in applying these standard axioms to cumulative payoffs, instead of incremental ones. Now we present our main representation result. All proofs are contained in the Appendix, and a sketch of the proof is provided in §4.

Theorem 1. *A preference $\succcurlyeq \subseteq \mathcal{A}^2$ satisfies A1-A6 if and only if there is a strictly increasing function $u : C \to \mathbb{R}$ and a non-increasing function $s : \mathcal{T} \to (0, 1]$, with $s(0) = 1$, such that*

$$a \succcurlyeq b \iff \sum_{t=0}^{T} [s(t) - s(t+1)] \sum_{x \in C} a(t, x) u(x) \geq \sum_{t=0}^{T} [s(t) - s(t+1)] \sum_{x \in C} b(t, x) u(x).$$

Moreover, a strictly increasing function $v : C \to \mathbb{R}$ and a non-increasing function $s' : \mathcal{T} \to (0, 1]$, with $s'(0) = 1$ satisfies $a \succcurlyeq b \iff \sum_{t=0}^{T} [s'(t) - s'(t+1)] \sum_{x \in C} a(t, x) v(x) \geq \sum_{t=0}^{T} [s'(t) - s'(t+1)] \sum_{x \in C} b(t, x) v(x)$ if and only if $s' = s$ and there are numbers $\alpha > 0$ and β such that

$$v(x) = \alpha u(x) + \beta.$$

Next, we translate the result to preferences over risky payoff streams to obtain the *discounted incremental utility* (DIU) model. For convenience, set $u(\sum_0^{-1}) = 0$.

Corollary 1. *By A0, if* (**??**) *represents* $\succcurlyeq \subseteq \mathcal{A}^2$, *then* $\succcurlyeq \subseteq \mathcal{X}^2$ *is represented by*

$$\begin{aligned} V_{\text{DIU}}(\mathbf{X}) &= \sum_{t=0}^{T} s(t)\, \mathbb{E} \left[u\left(\sum_{i=0}^{t} X_i \right) - u\left(\sum_{i=0}^{t-1} X_i \right) \right] \\ &= \mathbb{E}u(X_0) + \sum_{t=1}^{T} s(t)\, \mathbb{E} \left[u\left(\sum_{i=0}^{t} X_i \right) - u\left(\sum_{i=0}^{t-1} X_i \right) \right]. \end{aligned} \tag{1}$$

2.5 Time as Inherently Uncertain

We can heuristically derive (1) by combining EU with the interpretation of *time as inherently uncertain*. Suppose the individual lives under the fear that payoffs may be terminated at any time $t \in \mathcal{I}$, ending up with just the profit accrued at t, $\sum_{i=0}^{t} X_i$. For $t \in \mathcal{I}$, let $s(t)$ be the subjective probability of terminating at t or later, and conveniently set $s(T+1) = 0$. Thus, the probability of terminating precisely at t is $s(t) - s(t+1)$. Figure 2 views the risky payoff stream of Figure 1 under this two-stage lens. The first stage indicates the subjective probability of terminating at any $t \in \mathcal{I}$ and the second stage has the objective probabilities of profits conditional on termination. Thus, keeping in mind that $s(T+1) = 0$, the subjective expected utility of the risky payoff stream is

$$\begin{aligned} V_{\text{DIU}}(\mathbf{X}) &= \sum_{t=0}^{T} [s(t) - s(t+1)] \mathbb{E}u\left(\sum_{i=0}^{t} X_i \right), \\ &= s(T)\mathbb{E}u\left(\sum_{i=0}^{T} X_i \right) + \sum_{t=0}^{T-1} [s(t) - s(t+1)] \mathbb{E}u\left(\sum_{i=0}^{t} X_i \right) \end{aligned} \tag{2}$$

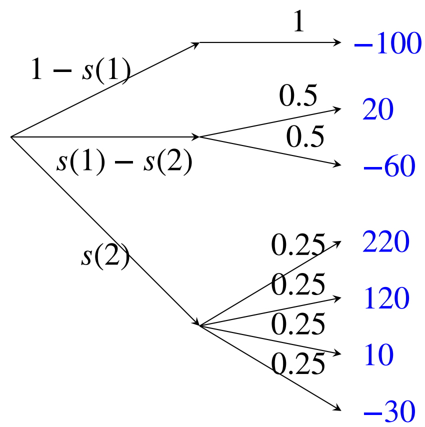
which is similar in form to (**??**), and an equivalent way to write (1).

2.6 Income Rates

Let $C(t)$ be cumulative income at time t, and $C'(t)$ the income rate; and assume u is differentiable. In line with the satiation model with full retention (Baucells and Zhao, 2020), DIU becomes

$$\begin{aligned} V_{\text{DIU}-rates}(\mathbf{X}) &= \mathbb{E}u(C(0)) + \mathbb{E} \int_0^T s(t)\, u'(C(t)) C'(t) dt \\ &= s(T)\mathbb{E}u(C(T)) + \mathbb{E} \int_0^T [-s'(t)] u\,(C(t))\, dt. \end{aligned}$$

Figure 2: Perception of a risky payoff stream when time is inherently uncertain.



The first expression is the analog of (1) for income rates. The second expression results from integration by parts, and it is the analog of (2). When $C(t)$ is a step function reflecting discrete lump payments, a continuity argument shows that $V_{\text{DIU}-rates}$ converges to the discrete version of V_{DIU} (see Baucells and Zhao (2020, p. 5704)).

3 Special Versions

Special versions of DIU contain the standard models as well as some novel forms. Clearly, if u is linear, then $V_{\text{DIU}} = \sum_{t=0}^{T} s(t)\mathbb{E}X_t$, and we get the *discounted expected payoffs*. We now explore other less trivial special cases of DIU.

3.1 Expected Utility of the Sum

If $s(T) \to 1$, then the patient individual does not think the stream will terminate prematurely, and DIU becomes the **Expected Utility of the sum**, $V_{\text{EU-SUM}}(\mathbf{X}) = \mathbb{E}u\left(\sum_{i=0}^{T} X_i\right)$. To illustrate the difference between DIU and EU-sum, let $X : \Omega \to \mathbb{R}_+$ be a positive risky payoff at time $t > 0$. Suppose that, to obtain X, one must put a deposit of $\delta > 0$ today, and get back the deposit at time t. Consistently with DIU, we expect individuals to be apprehensive of such deposit:

Pattern 1 (*the fear of being in the red*)**.** $(-\delta, 0) + (\delta + X, t) \prec (X, t)$.

DIU in Eq. (1) evaluates this opportunity as $s(t)\mathbb{E}u(X) + (1 - s(t))u(-\delta)$, i.e., the discounted expected utility of the final profit, plus a negative term expressing the fear of early termination, leading to a loss of the deposit. Increasing the deposit amount leaves the total profit unaltered (EU-sum is indifferent to δ), but makes the fear term more negative (DIU dislikes δ).

3.2 Discounted Kelly's criterion

Let $u(x) = \ln(C_{-1} + x)$, where $C_{-1} > 0$ is some initial capital the individual is willing to put at risk, and $C_t = C_{-1} + \sum_{i=0}^{t} X_i$ is the capital at time t. Then, the incremental utility $u(\sum_{i=0}^{t} X_i) - u(\sum_{i=0}^{t-1} X_i)$ happens to be log-return on capital, $\ln(C_t/C_{t-1})$, whose maximization produces the Kelly's criterion. Thus, DIU simply discounts the log-returns on capital, $V_{\text{DIU}}(\mathbf{X}) = \sum_{t=0}^{T} s(t)\,\mathbb{E}\ln(C_t/C_{t-1})$.

Discounted Kelly is different from the original Kelly criterion, $V_{\text{KELLY}}(\mathbf{X}) = s(T)\ln\left(C_T/C_0\right)$. To see this, apply (2) to see that

$$V_{\text{DIU}}(\mathbf{X}) = s(T)\ln\left(C_T/C_0\right) + \sum_{t=0}^{T-1}[s(t) - s(t+1)]\mathbb{E}\ln\left(C_t/C_0\right)$$

3.3 Red-averse utility

Let $u(x)$ possess a concave kink at some initial reference wealth level, e.g., the status quo, separating gains and losses. Subsequently, the incremental utility of a positive cash flow is always positive, but it is encoded as a much welcomed reduction of the loss if it offsets past losses. Similarly, the incremental utility of a negative payoff is always negative, but it is encoded as a mere gain reduction if the accumulated profit stays in the black, and as a painful loss if it turns (or keeps) past profits into the red.

The red-averse utility model explains very well the disposition effect in financial markets, depending on whether negative returns are encoded as a lesser gain or a larger loss, relative to some anchor (e.g., the purchase price). The individuals risk appetite increases when the current profit is negative (as the risk profile of the remaining payoffs uses a flatter portion of the value function). As soon as the profits recover, the risk profile of the remaining profits is centered at the reference point, and the kink induces a desire to avoid risk and sell. Here, we assume payoffs are not yet realized, which then encourages the reference point to stay at the purchase price (Imas, 2016). More generally, a prior payoff can alter whether a future payoffs yields cumulative gains of losses, hence influence future risk preferences and temporal trade-offs.

3.4 Relationship with Discounted Expected Utility

How does DIU compare with the traditional discounted expected utility (DEU) applied to money, $V_{\text{DEU}}(\mathbf{X}) = \sum_{t=0}^{T} s(t)\mathbb{E}u\left(X_t\right)$, often used as a benchmark to interpret experiments on time preferences (Rubinstein, 2003; Andreoni and Sprenger, 2012)? DIU does not contain DEU, except for the trivial cases of one-time lotteries, or linear utility. DEU, however, can be obtained from a modified

Table 2: Evaluation of one or two plays of $X = (350, -200; 0.5, 0.5)$ using Eq. (1) with $u(x) = x \cdot 1_{x\geq 0} + 2.25x \cdot 1_{x<0}$ and $s(1) = 0.9$.

$u(X, 0) =$	$\frac{1}{2}350 - \frac{1}{2}200 \cdot 2.25$	$= -50$
$u(X + X, 0) =$	$\frac{1}{4}700 + \frac{1}{2}150 - \frac{1}{4}400 \cdot 2.25$	$= 25$
$u(X + X, 1) =$	$s(1)u(X + X, 0)$	$= 22.5$
$u((X, 0) + (X, 1)) =$	$(1 - s(1))u(X, 0) + s(1)u(X + X, 0)$	$= 17.5$

version of DIU where the reference point is dynamically reset at the status quo in every period.[6]

Using experimental data, Cheung (2019) calibrated both DEU and DIU, and concluded [p.514] that DIU has a superior log-likelihood with the same number of parameters. DIU also outperforms the linear model in terms of AIC and BIC. We now explain how DIU outperforms DEU in predicting attitudes towards sequential play and preference for temporal hedging.

3.5 Back to Samuelson's Paradox

How individuals react to the repeated play of a lottery has been crucial in the understanding of expected utility (Samuelson, 1963). Under the lens of DIU, we bring the time dimension and explore how individuals react to different timings of repeated play. We will show that DIU is more nuanced than DEU, in that it exhibits some myopic loss aversion, but not to the point of rejecting the repeated play of a favorable lottery.

Table 2 applies the red-averse version of DIU using one or two repetitions of the mixed lottery $X = (350, -200; 0.5, 0.5)$, assuming a piece-wise linear utility with loss aversion 2.25. We observe that the single play is rejected (row 1), whereas the simultaneous repetition of two plays today is accepted (row 2). Delaying this simultaneous repetition (row 3) makes it a bit less attractive due to discounting, but the total utility cannot switch signs. The sequential play of one lottery today and another tomorrow (row 4) would also be acceptable. The delayed simultaneous play (row 3), however, is preferable to sequential play (row 4). This is because if the first play comes up as a loss, then sequential play leaves the decision maker in the red for one period, whereas delayed simultaneous offsets the losses immediately. A simple condition summarizes the predicted patterns.

[6] Let $u_t(x) = u(x - r_t)$, where u is a value function and $r_t = \sum_{i=0}^{t-1} X_i$ is such a dynamic reference point. In a financial investing context, this reference point updating would be realistic if the decision maker sells its position and realizes the profits in each period (Imas, 2016). Then, the incremental utility becomes the utility of the current payoff, as follows:

$$u_t\left(\sum_{i=0}^{t} X_i\right) - u_t\left(\sum_{i=0}^{t-1} X_i\right) = u(r_t + X_t - r_t) - u(r_t - r_t) = u(X_t).$$

In short, discounted incremental utility plus full reference point adaptation equals DEU. Yet, because u_t now depends on time via r_t, this adaptive model violates risk and time independence (A5), hence falls outside our axiomatization.

Pattern 2 (*attitudes towards sequential play*)**.** *Let* $X : \Omega \to \mathbb{R}$ *and* $t > 0$ *be such that* $(X+X,0) \succ 0 \succ (X,0)$*. Then, a)* $(X,0)+(X,t) \succ 0$ *is possible, and b)* $(X+X,t) \succ (X,0)+(X,t)$*.*

While Pattern 2b is compatible with DEU, Pattern 2a is not, as DEU predicts that the more sequential repetitions, the worse: $(X,0) \prec 0$ implies $\sum(X,t_i) \prec 0$.

Are observed preferences compatible with Pattern 2? In preparation for a class discussion at the Darden School of Business, 140 MBA students enrolled in an elective during the Fall of 2019 and 2020 had to complete a survey. While payoffs are hypothetical, students are well aware that their responses play a key role in their learning experience. They were presented with the opportunity of playing 10 repetitions of $X = (\$300, -\$200; 0.5, 0.5)$ with three different timings: simultaneous play today, simultaneous play in 10 weeks, and sequential play spread over 10 weeks.

Table 3: Play of ten repetitions of $X = (\$300, -\$200; 0.5, 0.5)$ with various timings. The notation is analogous to that in Table 2.

	Question		Answer [N each side]	p-value†
(i)	$(X,0)$	vs. 0	36% vs. 64% $[48, 84]$	0.0012
(ii)	$\left(\sum_{i=1}^{10} X, 0\right)$	vs. 0	84% vs. 16% $[117, 23]$	0.0000
(iii)	$\left(\sum_{i=1}^{10} X, 10\right)$	vs. 0	76% vs. 24% $[106, 34]$	0.0000
(iv)	$\sum_{i=1}^{10}(X,i)$	vs. 0	60% vs. 40% $[84, 56]$	0.0140
(v)	$\left(\sum_{i=1}^{10} X, 10\right)$	vs. $\sum_{i=1}^{10}(X,i)$	61% vs. 39% $[86, 54]$	0.0051

†Binomial two-tail test, null hypothesis 50%.

Table 3 shows the results. First, we confirm Samuelson's paradox, namely, the majority (i) rejects a single play,[7] but (ii) accepts 10 repetitions. Most also (iii) accept 10 simultaneous repetitions played 10 weeks from now, but with slightly less enthusiasm. The prospect of 10 sequential repetitions over 10 weeks is attractive, but much less so (iv). When asked directly, (v) the majority prefers delayed simultaneous play to sequential play. Thus, these MBA's are fully consistent with Pattern 2.

Note that while the DIU prediction is fully consistent with the presented evidence, it is also much more natural than the DEU prediction. According to DEU, the decision maker who rejects a single play, equally dislikes each of its sequential repetitions, making the whole thing worse the more repetitions there are. On the other hand, consistently with the Samuelson example, DEU predicts that in the case of simultaneous play, more repetitions make the prospect more attractive. Obviously, preferences cannot be so divergent, especially because the boundary between the sequential and the simultaneous play is not very sharp, and depends on how we define time periods. DIU, on the other

[7] In an earlier survey, these same students were asked for the minimum gain G that would make an equal chance of $L = -\$500$ and G acceptable. Most students [64%] required $G > 1.5 \cdot |L|$.

hand, captures the intuition quite well. Note that for $0 \leq t_1 < t_2 < ... < t_n$ by (1eq.) the advantage of sequential over delayed simultaneous is given by

$$V\left(\sum_{i=1}^{n}(X, t_i)\right) - V\left(\sum_{i=1}^{n} X, t_n\right) = \sum_{i=1}^{n-1}[s(t_i) - s(t_{i+1})]V\left(\sum_{t=1}^{i} X, 0\right), \tag{3}$$

where $V\left(\sum_{t=1}^{i} X, 0\right)$ is the evaluation of i simultaneous repetitions today. If all time periods are close to each other, the decision maker is unlikely to believe termination occurs before t_n, and hence the difference in (3) will be close to null. If on the other hand, the decision maker believes termination is very likely/is very impatient, then sequential play is preferable to delayed simultaneous if all these terms are positive (any number of repetitions is favorable), and the opposite if negative (due to loss aversion).

3.6 Serial Correlation and Temporal Hedging

Consider a 50:50 lottery played today, and played again at some future time with the odds tilted so that the same outcome may repeat with more or less than 50%. Specifically, for $x > 0 \geq y$, $t > 0$, and $0 \leq \theta \leq 1$, let

$$\mathbf{X}^{\theta} = ((x, y; 0.5, 0.5), 0) + \begin{cases} ((x, y; \theta, 1-\theta), t) \text{ if } (x, 0) \text{ realizes,} \\ ((x, y; 1-\theta, \theta), t) \text{ if } (y, 0) \text{ realizes.} \end{cases} \tag{4}$$

Negative serial correlation, $\theta < 0.5$, offers a form of temporal hedging. Indeed, if $\theta' < \theta$, then the final profit in $\mathbf{X}^{\theta'}$ is a mean preserving contraction of that in $\mathbf{X}^{\theta}$. We expect most individuals to be risk averse, hence like negative correlation, and dislike positive correlation.

Pattern 3 (*preference for temporal hedging*)**.** *If $\theta' < \theta$, then $\mathbf{X}^{\theta'} \succ \mathbf{X}^{\theta}$.*

Using $x = 350$ and $y = -200$ as in Table 2, the uncorrelated case with $\theta = 1/2$ yields a final profit of $(700, 150, 400)$ with probabilities $(1/4, 1/2, 1/4)$, setting $\theta = 2/3$ (positive serial correlation) changes the probabilities to $(1/3, 1/3, 1/3)$ (row 4), and setting $\theta = 1/3$ (negative serial correlation) changes the probabilities to $(1/6, 2/3, 1/6)$. Using the same parameters in Table 2, the utility of these three prospects is 17.5, -20, and 55, respectively. More generally, we verify that, *under DIU, a preference for temporal hedging holds if and only if u is mid-point concave, i.e.,* $u(x+y) > [u(2x) + u(2y)]/2$. In contrast, DEU is insensitive to serial correlation.

4 Outline of the proofs

Theorem 1: We follow Fishburn (1970, Thm 13.2) subjective expected utility result, after reinterpreting its two stages. For us, the 'horse race' determines the termination time $t \in \mathcal{I}$, and the 'roulette wheel' pays the profits accrued by then, a_t, following the distribution $\mathbb{P}$. Indeed, A1-A3 are standard vNM axioms applied to the mixture space $\mathcal{A}$. Axiom A5 is nothing else than the state-independence axiom, which is key in separating subjective beliefs from tastes over outcomes, and A6 strengthens their non-triviality axiom to ensure that not only the set $\mathcal{I}$ but also all its nonempty subsets are non-null.

5 Conclusions

Our contribution in this paper is two-fold. On the normative side, our starting point was an individual who cares about profits. When profits accrue over time, such individual should care not only about final profits, but also on how soon these profits accrue. The standard axioms of expected utility, together with impatience and risk&time independence, then produce the incremental discounted utility model, whose form was advanced by Bell (1974). The model is normative when the utility is a function of wealth and discounting is exponential, and can take intriguing new forms such as the discounted Kelly's criterion. Discounted incremental utility can be heuristically derived by treating time as inherently uncertain (Baucells and Heukamp, 2012; Blavatskyy, 2016).

Discounted incremental utility is behaviorally appealing when the utility exhibits a concave kink at some reference point. There, a negative payoff does not 'count double' when it reduces previous positive profits, whereas a positive payoff can 'count double' when it reduces a past loss. This specification predicts that when confronted with different payoff streams having the same cumulative total at the end, individuals will prefer those that stay in the red for a shorter time, e.g., prefer investments with shorter payback periods (Graham and Harvey, 2001).

To further increase the descriptive appeal of discounted incremental utility, we add range distortion effects. Range principles, with a long tradition in psychology, and recently modeled for risk by Kontek and Lewandowski (2017) and Baucells, Lewandowski and Kontek (2020), offer an avenue more promising than rank principles when it comes to adding the time dimension. Here, our contribution tackles a fundamental challenge: The two leading behavioral models for risk and time possess narrow and non-overlapping domains. Indeed, Prospect theory applies to one-time lotteries (Tversky and Kahneman, 1992); whereas hyperbolic discounting applies to payoff streams under certainty (Halevy, 2008; Blavatskyy, 2016). Thus, it is unclear how these two paradigms are to be merged. Something as simple as a two coin bets played at different times cannot be currently handled well. Applying prospect theory to say the discounted payoffs would not properly take into account the feeling of loss triggered if the first coin delivers a negative payoff, a feeling that DIU sustains at least until the second coin is played. The alternative, discounting the utility of each payoff, induces unrealistic degrees of loss aversion, and is difficult to square with skewness seeking (e.g., via non-linear probability weighting). We believe Range-DIU provides a fertile ground to successfully address this challenge.

To our knowledge, Range-DIU is the first behavioral model that jointly captures a wide array of observed phenomena in the broad domain of risky payoff streams. The model is uniquely positioned to also make predictions involving both risk and time, hence provide a frame to interpret the results of experiments in this immense and mostly unexplored domain.

A Proofs

We begin with some preliminary definitions and results. For any convex set K, a function $U : K \to \mathbb{R}$ is affine if for any $a, b \in K$ and $\alpha \in [0,1]$, $U(\alpha a + (1-\alpha)b) = \alpha U(a) + (1-\alpha)U(b)$. We say that a subset I of $\mathcal{I}$ is *null* if $a \sim b$ holds for any pair of profiles a, b such that $a_t = b_t$ for all $t \notin I$. The following vNM representation for mixtures spaces is standard, and hence omit the proof.

Theorem 2 (**vNM theorem for acts**). *A binary relation $\succcurlyeq$ on $\Delta(C)^{\mathcal{T}}$ satisfies A1–A3 if and only if there exist a function $u : \mathcal{T} \times C \to \mathbb{R}$ such that for all $a, b \in \mathcal{A}$*

$$a \succcurlyeq b \iff \sum_{t \in \mathcal{T}} \sum_{x \in C} a(t,x)u(t,x) \geq \sum_{t \in \mathcal{T}} \sum_{x \in C} b(t,x)u(t,x). \tag{5}$$

Moreover, a function $v : \mathcal{T} \times C \to \mathbb{R}$ satisfies $a \succcurlyeq b \iff \sum_{t \in \mathcal{T}} \sum_{x \in C} a(t,x)v(t,x) \geq \sum_{t \in \mathcal{T}} \sum_{x \in C} b(t,x)v(t,x)$ if and only if there are numbers $\alpha > 0$ and $\beta(t)$ for $t \in \mathcal{T}$ such that

$$v(t,x) = \alpha u(t,x) + \beta(t) \quad \textit{for all} \quad (t,x) \in \mathcal{T} \times C.$$

Theorem 3 (**vNM theorem for lotteries**). *A binary relation $\succcurlyeq$ on $\Delta(C)$ satisfies A1–A3 if and only if there exists a function $u : C \to \mathbb{R}$ such that*

$$p \succcurlyeq q \iff \sum_{x \in C} p(x)u(x) \geq \sum_{x \in C} q(x)u(x). \tag{6}$$

Moreover, a function $v : C \to \mathbb{R}$ satisfies $p \succcurlyeq q \iff \sum_{x \in C} p(x)v(x) \geq \sum_{x \in C} q(x)v(x)$ if and only if there are numbers $\alpha > 0$ and β such that

$$v(x) = \alpha u(x) + \beta \quad x \in C.$$

Proof of Theorem 1. Necessity of the axioms is straightforward. We prove sufficiency by adapting Fishburn (1970, Thm 13.2)'s proof to our time setup. Assume that A1–A5 hold. Then according to Theorem 2 there exist a function $u : \mathcal{T} \times C \to \mathbb{R}$ such that (5) holds for all $a, b \in \mathcal{A}$. It is straightforward to verify that $t \in \mathcal{T}$ is null if and only if $u(t, \cdot)$ is constant. In view of the uniqueness part of Theorem 2 we may assume that this constant is zero. Axiom A4 implies that $\mathcal{T}$ is nonnull. We will show that for any $t' \in \mathcal{T}$ such that $\{t \in \mathcal{T} : t \geq t'\}$ is a nonnull event, the function $\sum_{t=t'}^{T} u(t, \cdot)$ represents $\succcurlyeq$ on $\Delta(C)$. Suppose that for some $p, q \in \Delta(C)$, $\sum_{x \in C} p(x) \sum_{t=t'}^{T} u(t,x) >$

$\sum_{x\in C} q(x) \sum_{t=t'}^{T} u(t,x)$. Let $a = (p, t')$ and $b = (q, t')$. Then

$$\sum_{t\in\mathcal{T}} \sum_{x\in C} a(t,x)u(t,x) > \sum_{t\in\mathcal{T}} \sum_{x\in C} b(t,x)u(t,x).$$

Applying (5), we obtain $a \succ b$, which, in view of A5, implies $p \succ q$.

If for some $p, q \in \Delta(C)$, $\sum_{x\in C} p(x) \sum_{t=t'}^{T} u(t,x) = \sum_{x\in C} q(x) \sum_{t=t'}^{T} u(t,x)$, we define $a(t,\cdot) = p$ and $b(t,\cdot) = q$ for all $t \in \mathcal{T}$. Then

$$\sum_{t\in\mathcal{T}} \sum_{x\in C} a(t,x)u(t,x) = \sum_{t\in\mathcal{T}} \sum_{x\in C} b(t,x)u(t,x),$$

which, in view of (5), implies $a \sim b$. By the definition of $\succcurlyeq$ on $\Delta(C)$, the latter implies $p \sim q$.

We know that the event $\{t \in \mathcal{T} : t \geq 0\} = \mathcal{T}$ is nonnull. Take $t' \in \mathcal{T}$, $t' \neq 0$ such that $\{t \in \mathcal{T} : t \geq t'\}$ is nonnull. By the above argument, we know that both $\sum_{t=t'}^{T} u(t,\cdot)$ and $\sum_{t=0}^{T} u(t,\cdot)$ represent $\succcurlyeq$ on $\Delta(C)$. Thus, by the uniqueness part of Theorem 3, we conclude that there are numbers $\alpha(t) > 0$ and $\beta(t)$, $t \in \mathcal{T}$ such that

$$\sum_{t=t'}^{T} u(t,x) = \alpha(t) \sum_{t=0}^{T} u(t,x) + \beta(t).$$

We define $u(\cdot) := \sum_{t=0}^{T} u(t,\cdot)$. By the uniqueness part of Theorem 2 we can choose $\beta(t) = 0$, $t \in \mathcal{I}$. We also define $\alpha(0) = 1$. Hence, for $t < T$, $\sum_{i=t}^{T} u(i,\cdot) = \alpha(t)u(\cdot)$ and $\sum_{i=t+1}^{T} u(i,\cdot) = \alpha(t+1)u(\cdot)$. Subtracting the former from the latter yields

$$u(t,\cdot) = [\alpha(t) - \alpha(t+1)]u(\cdot) \quad \text{for} t < T, \tag{7}$$

and

$$u(T,\cdot) = \alpha(T)u(\cdot). \tag{8}$$

Finally, we define $s(t) = \frac{\alpha(t)}{1-\alpha(T)}$, $t \in \mathcal{T}$. A6 implies that $s(t) \leq s(t')$ for $t, t' \in \mathcal{T}$, $t' < t$. Indeed, let $x \in C$, $x > 0$, then by A6 $(\delta_x, t') \succcurlyeq (\delta_x, t)$ for $t, t' \in \mathcal{T}$, $t' < t$. This, In view of (7),(8) and (5), implies the following

$$u(x) \sum_{s\geq t'} (\alpha(s) - \alpha(s+1)) \geq u(x) \sum_{s\geq t} (\alpha(s) - \alpha(s+1)),$$

which in turn yields $\alpha(t') \geq \alpha(t)$ or $s(t') \geq s(t)$. □

Proof of Corollary 1. Let $\mathbf{X} = (X_t : \Omega \to \mathbb{R})_{t\in\mathcal{I}}$. Because all X_t are finite-valued random variables, $a_t(\mathbf{X})$ has finite support, hence $a(\mathbf{X}) \in \mathcal{A}$. Note that $\sum_{x\in\mathbb{R}} a_t(\mathbf{X})(x)u(x) = \mathbb{E}u\left(\sum_{i=0}^{t} X_i\right)$ for each $t \in \mathcal{I}$, so that $U(a)$ in (**??**) equals $u(\mathbf{X})$ in (1eq.), which is equivalent to (1). Hence, by A0, if $U(\cdot)$ represents $\succcurlyeq$, then $u(\cdot)$ must represent $\succcurlyeq'$. $\square$

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